QUASITORIC MANIFOLDS, ROOT SYSTEMS AND J-CONSTRUCTIONS OF POLYTOPES

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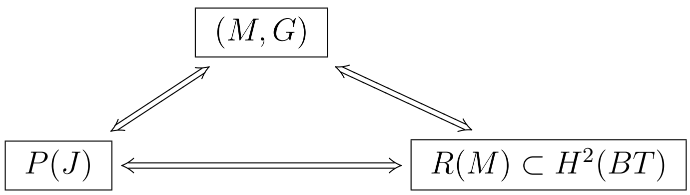
1. Introduction

This article is a research announcement of the progress paper [Ku3].

Let (M, T) be a quasitoric manifold over P. In this article, we assume that there is an extended G-action of (M, T), say (M, G). See [Ku1] about the extended action and the definition of quasitoric manifolds. The purpose of this article is to introduce the following relation among geometry, algebra and combinatorics.



Here, $P(J)$ is the J-construction of a polytope P which is introduced by Bahri-Bendersky-Cohen-Gitler in [BBCG1] (see Section 3), and $R(M)$ is the root systems of a quasitoric manifold introduced by the author and Masuda in [KM] (see Section 2). This relation can be summarized as the following theorem (also see Theorem 4.1):

Theorem 1.1. *If $R(M)$ has a simple factor $A_j (= \langle \alpha_1, \ldots, \alpha_{j+1} \,|p^*(\alpha_i) = \tau_i - \tau_{i+1} \rangle)$ (i.e., $R(M) \neq \emptyset$), then there is an $(n-j)$-dim simple polytope P' such that one of the following holds:*

(1) $P \simeq P' \times \Delta^j$;
(2) $P \simeq P'(j+1, 1, \ldots, 1) (=: P'(J))$,

where Δ^j is the j-dimensional simplex and $P'(J)$ is the J-construction of polytope.

In [BBCG1], the J-construction of P is defined as purely combinatorial way. So this theorem may be regarded as the geometric meaning of the J-construction of P in the context of toric topology.

As a corollary of this result, we also have

Corollary 1.2. There is an extended (M, G) of (M, T) if and only if $R(M) \neq \emptyset$.

2. Root systems of quasitoric manifolds

We first introduce the definition of *root systems* of quasitoric manifolds (see [KM] for details).

To do that we need to recall the generators in the 2nd degree equivariant cohomology of quasitoric manifolds. Let $\pi : M \to P$ be the orbit projection of quasitoric manifold and $M_i := \pi^{-1}(F_i)$ be the characteristic submanifold corresponding to the facet F_i, $i = 1, \ldots, m$. Set the equivariant Thom class of M_i as τ_i (for the fixed omniorientation of M). Then, the following isomorphism holds:

$$H^2_T(M) = \mathbb{Z}\tau_1 \oplus \cdots \oplus \mathbb{Z}\tau_m,$$

that is the 2nd degree equivariant cohomology is generated by the equivariant Thom classes of characteristic submanifolds. Let $p : ET \times_T M \to BT$ be the projection of the Borel construction of M. Then, the induced injective map $p^* : H^2(BT) \to H^2_T(M)$ is defined as

$$p^*(\alpha) = \sum_{i=1}^{m} \langle \alpha, \lambda(F_i) \rangle \tau_i,$$

where $\lambda : \{F_1, \ldots, F_m\} \to \mathfrak{t}_{\mathbb{Z}}$ is the characteristic function on P and $\langle , \rangle$ is the evaluation of $H^2(BT) \simeq \mathfrak{t}^*_{\mathbb{Z}}$ and $H_2(BT) \simeq \mathfrak{t}_{\mathbb{Z}}$. The root systems of a quasitoric manifold can be defined as follows:

Definition 2.1. $R(M) := \{\alpha \in H^2(BT) \mid p^*(\alpha) = \tau_i - \tau_j\}$ is called a *root systems of a quasitoric manifold.*

We proved the following result in the previous paper [KM]:

Theorem 2.2. *If there is an extended (M, G) of a quasitoric (M, T), then*

$$R(G) \subset R(M),$$

where $R(G)$ is the root systems of G.

Note that the following corollary holds:

Corollary 2.3. If $R(M) = \emptyset$, then there is no extended action (M, G) of a quasitoric (M, T).

This shows that the root systems of a quasitoric manifold is an invariant of the existence of an extended actions. Therefore, the following question is the natural question: Is $R(M)$ the complete invariant of the existence of an extended actions? Corollary 1.2 answers to this question.

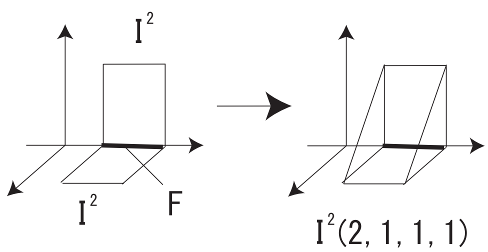
3. J-construction of polytopes

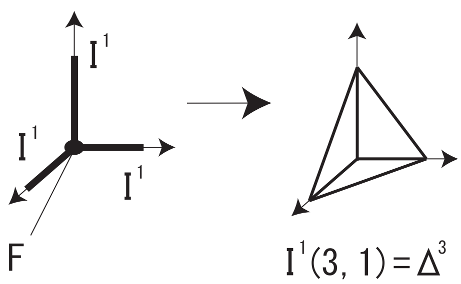
We next introduce the J-construction of a simple polytope. Note that in [BBCG1] the J-construction is defined for the simplicial complex. The following definition is the dual of the definition in [BBCG1]. Let P^n be a simple convex polytope with m-facets. Let $J = (j, 1, \ldots, 1) \in \mathbb{N}^m$.

Definition 3.1. The *J-construction of polytope*, denoted by $P(J)$, is defined by the following way:

(1) Fix a facet $F \subset P$ and take j copies of P say $P_1, \ldots, P_j$;
(2) Embed $P_i \subset \mathbb{R}^{n-1} \times [0, \infty)$ such that $F = P_i \cap \mathbb{R}^{n-1}$, then we have $P_1 \cup \cdots \cup P_j \subset \mathbb{R}^{n-1} \times [0, \infty)^j$ and $P_1 \cap \cdots \cap P_j = F \subset \mathbb{R}^{n-1}$;
(3) $P(J) := \mathrm{Conv}(P_1 \cup \cdots \cup P_j)$.

The following figures show two examples of J-constructions of polytopes.





Remark 3.2. When $j = 2$, this is also called a *(simplicial) wedge operation* (Ewald). For $J = (j_1, \ldots, j_m)$, we can also define J-construction by the iteration of this construction (Bahri-Bendersky-Cohen-Gitler).

4. Main theorem and some conclusions

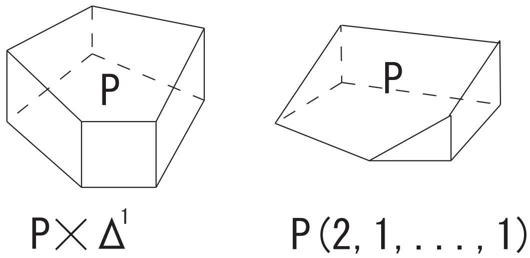
In [Ku3], we prove the following theorem:

Theorem 4.1. *If $R(M)$ has a simple factor $A_j(=\langle \alpha_1, \ldots, \alpha_{j+1} \,|p^*(\alpha_i) = \tau_i - \tau_{i+1}\rangle)$ (i.e., $R(M) \neq \emptyset$), then there is an $(n-j)$-dim simple polytope P' such that one of the following holds:*

(1) $P \simeq P' \times \Delta^j$;
(2) $P \simeq P'(j+1, 1, \ldots, 1)(=: P'(J))$,

where Δ^j is the j-dimensional simplex and $P'(J)$ is the J-construction of polytope.

For example, if $R(M) = A_1$ $(j = 1)$, then M/T is one of the following type of polytopes: Each case satisfies $p^*(\alpha) = \tau_1 - \tau_2$, where τ_i is a Thom class corresponds to up and down facets P.



Remark 4.2. The left polytope may be regarded as a blow-up of the right polytope (also see [Ku2]). This also gives another combinatorial meaning of the J-construction of P, i.e., if we blow-up $P(J)$ along some face, then it becomes the product with simplicies. This might be not known fact in combinatorics but it is the known fact in geometry (see [Wi1]).

Finally, by using theorem of [BBCG2], we also have the following proposition.

Proposition 4.3. If $R(M)$ has an A_j simple factor and $P = P' \times \Delta^j$ then

$$M \simeq S^{2j+1} \times_{S^1} M(P', \lambda') \simeq (S^{2j+1} \times \mathcal{Z}_{P'})/H(= \mathcal{Z}/H),$$

where $M(P', \lambda')$ is the quasitoric of (P', λ').

If $R(M)$ has an A_j simple factor and $P = P'(j+1, 1, \ldots, 1)$ then

$$M \simeq \mathcal{Z}(K_{P'}, (\underline{D}^{2J}, \underline{S}^{2J-1}))/\ker \lambda(= \mathcal{Z}/H),$$

where $(\underline{D}^{2J}, \underline{S}^{2J-1}) = \{(D^{2j_i}, S^{2j_i-1})\}_{i=1}^m$ $(j_1 = j+1, j_i = 1 (i \neq 1))$.

In summary, we have that

Corollary 4.4. Every extended action (M, G) of quasitoric (M, T) is induced from the $\widetilde{G}$-action on the polyhedral product, where $\widetilde{G}$ is the finite covering of G.

This is the generalization of [Ku1, Theorem11.2].

The details of the facts and notations in this article will be appeared in [Ku3].

Acknowledgment

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