A remark on torus graph with root systems of type A

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1. Introduction

In the previous paper [**KuMa**], we define a root system on a torus manifold, and characterize extended actions of torus manifolds. Due to the work of Maeda-Masuda-Panov [**MMP**], there is a combinatorial counterpart of torus manifold, called a *torus graph* $(\Gamma, \mathcal{A})$. Here, $\Gamma = (V(\Gamma), E(\Gamma))$ is an abstract n-valent graph and $\mathcal{A} : E(\Gamma) \to H^2(BT^n)$ is a label on edges, called an *axial function*. Therefore, we can also define *root systems* on torus graphs like torus manifolds. In this article, we characterize the torus graph with root systems of type A combinatorially.

2. Root systems of type A of torus graphs

Let $(\Gamma, \mathcal{A})$ be a torus graph, and $H_T^*(\Gamma, \mathcal{A})$ be its *graph equivariant cohomology*, i.e.,

$$H_T^*(\Gamma, \mathcal{A}) := \{f : V(\Gamma) \to H^*(BT^n) \mid f(p) \equiv f(q) \mod \mathcal{A}(e) \text{ for } i(e) = p, t(e) = q\},$$

where $i(e)$ (resp. $t(e)$) is the initial (resp. terminal) vertex of $e \in E(\Gamma)$. Then, we can define the following injective homomorphism

$$\varphi : H^*(BT^n) \to H_T^*(\Gamma, \mathcal{A})$$

by

$$\varphi(\alpha) := \alpha,$$

where $\alpha : V(\Gamma) \to H^*(BT^n)$ is the constant map, i.e., $\alpha(p) = \alpha$ for all $p \in V(\Gamma)$. Then, with the method similar to define a root system of type A of torus manifold in [**KuMa**], we can define a *root system of type A* on torus graph as follows.

Definition 2.1. We call the set $R(\Gamma, \mathcal{A}) \subset H^2(BT^n)$ a *root system of type A* of a torus graph $(\Gamma, \mathcal{A})$ if $\alpha \in R(\Gamma, \mathcal{A})$ then $-\alpha \in R(\Gamma, \mathcal{A})$ and $\varphi(\alpha) = \tau_i - \tau_j$ for some Thom classes τ_i and τ_j of $(n-1)$-valent torus subgraphs Γ_i and Γ_j.

Proposition 2.2. The above $R(\Gamma, \mathcal{A})$ satisfies the axiom of root systems in [**Hu**] with respect to the inner product of $H^2(BT^n)$ defined by the paring with $H_2(BT^n)$ (see [**KuMa**]).

Let $\Delta(\Gamma, \mathcal{A}) = \{\alpha_1, \ldots, \alpha_\ell\}$ be a simple root of $R(\Gamma, \mathcal{A})$. If there exists a string $\tau_1, \ldots, \tau_{\ell+1}$ of Thom classes such that $\varphi(\alpha_i) = \tau_i - \tau_{i+1}$ for all $i = 1, \ldots, \ell$, then $R(\Gamma, \mathcal{A})$ is called an *irreducible*.

3. Main theorem

In order to state the main theorem, we need to prepare some notations.

3.1. Fibration of torus graphs. We first recall the fibration of torus graphs (also see [**GSZ**]).

Let Γ and B be connected graphs and $\rho : \Gamma \to B$ be a morphism of graphs. Hence ρ is a map from the vertices of Γ to the vertices of B such that if $pq \in E(\Gamma)$ then either $\rho(p) = \rho(q)$ or else $\rho(p)\rho(q) \in E(B)$. If $pq \in E(\Gamma)$ and $\rho(p) = \rho(q)$ then we will say that the edge pq is *vertical*, and if $\rho(p)\rho(q) \in E(B)$ then we will say that the edge pq is *horizontal*. For a vertex $q \in V(\Gamma)$, let $E_q^{\perp}(\Gamma)$ be the set of vertical edges with initial vertex q, and let $H_q(\Gamma)$ be the set of horizontal edges with initial vertex q. Then $E_q(\Gamma) = E_q^{\perp}(\Gamma) \cup H_q(\Gamma)$ and ρ induces canonically a map

$$(d\rho)_q : H_q(\Gamma) \to E_{\rho(q)}(B)$$

from the horizontal edges at q to the edges of B with initial vertex $\rho(q)$: if $qq' \in H_q(\Gamma)$, then $(d\rho)_q(qq') = \rho(q)\rho(q')$.

Definition 3.1. The morphism of graphs $\rho : \Gamma \to B$ is a *fibration* of graphs if for every vertex q of Γ, the map $(d\rho)_q : H_q(\Gamma) \to E_{\rho(q)}(B)$ is bijective.

Let us define the fibration of torus graphs.

Definition 3.2. Let $(\Gamma, \mathcal{A})$ and $(B, \mathcal{A}_B)$ be torus graphs. A morphism $\rho : (\Gamma, \mathcal{A}) \to (B, \mathcal{A}_B)$ is a *fibration* of torus graphs, if it satisfies the following conditions:

(1) $\rho : \Gamma \to B$ is a fibration of graphs;
(2) If e is an edge of B and $\widetilde{e}$ is any lift of e, then $\mathcal{A}(\widetilde{e}) = \mathcal{A}_B(e)$.

Comparing with the definition of GKM-fibrations in [**GSZ**] (also see [**Ku**]), we do not need to assume the compatible conditions of connections. This is because the connections of torus graphs are uniquely determined. In particular, we have the following proposition.

Proposition 3.3. Let $\rho : (\Gamma, \mathcal{A}) \to (B, \mathcal{A}_B)$ be a fibration of torus graphs. Assume that Γ is n-valent and B is ℓ-valent. Then, for all $p \in V(B)$, $\rho^{-1}(p)$ is an $(n-\ell)$-valent torus subgraph of Γ.

3.2. Blow-up of torus graphs. We next introduce a blow-up of a torus graph (see [**MMP**]).

Let $(\Gamma', \mathcal{A}')$ be an $(n-\ell)$-valent torus subgraph of the n-valent GKM graph $(\Gamma, \mathcal{A})$. Then, the cardinality of the normal edges $N_p(\Gamma')$ is exactly ℓ; therefore, we may denote $N_p(\Gamma') = \{pp'_1, \ldots, pp'_\ell\}$.

The *blow-up* of Γ along Γ', denoted $\widetilde{\Gamma} = (V(\widetilde{\Gamma}), E(\widetilde{\Gamma}))$, is defined as follows. The vertex set is defined as $V(\widetilde{\Gamma}) = (V(\Gamma) - V(\Gamma')) \cup V(\Gamma')^\ell$, where $V(\Gamma')^\ell = V(\Gamma') \times \cdots \times V(\Gamma')$

(ℓ times Cartesian product), i.e., the vertex $p \in V(\Gamma') \subset V(\Gamma)$ is replaced by ℓ vertices $\widetilde{p}_1, \ldots, \widetilde{p}_\ell$. It is convenient to regard those points as chosen close to p on edges from $N_p(\Gamma') = \{pp'_1, \ldots, pp'_\ell\}$, i.e., $\widetilde{p}_i \in pp'_i$. Then the edges and the corresponding values of the axial function $\widetilde{\mathcal{A}} : E(\widetilde{\Gamma}) \to H^2(BT)$ are defined as follows:

(1) $\widetilde{p}_i\widetilde{p}_j \in E(\widetilde{\Gamma})$ for every $p \in V(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_i\widetilde{p}_j) = \mathcal{A}(pp'_j) - \mathcal{A}(pp'_i)$;
(2) $\widetilde{p}_i\widetilde{q}_i \in E(\widetilde{\Gamma})$ if $pq \in E(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_i\widetilde{q}_i) = \mathcal{A}(pq)$;
(3) $\widetilde{p}_ip'_i \in E(\widetilde{\Gamma})$ for every $p \in V(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_ip'_i) = \mathcal{A}(pp'_i)$;
(4) edges "coming from Γ", that is, $pq \in E(\Gamma)$ such that $p, q \notin V(\Gamma')$; $\widetilde{\mathcal{A}}(pq) = \mathcal{A}(pq)$.

Combinatorially, this operation is nothing but the gluing of $\Gamma' \times K_{\ell+1}$ along the subgraph $\Gamma' \subset \Gamma$, where $K_{\ell+1}$ is the complete graph with $(\ell+1)$-vertices, i.e., $V(K_{\ell+1}) = \{p_0, \ldots, p_\ell\}$, $E(K_{\ell+1}) = \{p_ip_j \mid i \neq j\}$.

The following proposition is straightforward.

Proposition 3.4. Let $(\Gamma, \mathcal{A})$ be an n-valent torus graph and $(\Gamma', \mathcal{A}')$ be a torus subgraph. Then, its blow-up $(\widetilde{\Gamma}, \widetilde{\mathcal{A}})$ along $(\Gamma', \mathcal{A}')$ is an n-valent torus graph. Moreover, there is the natural morphism from $(\widetilde{\Gamma}, \widetilde{\mathcal{A}})$ to $(\Gamma, \mathcal{A})$.

3.3. Main theorem.

The main theorem can be stated as follows:

Theorem 3.5. *Let $(\Gamma, \mathcal{A})$ be a torus graph. Suppose that there exists an irreducible non-empty root system of type A, say $R(\Gamma, \mathcal{A})$. Choose its simple root as $\Delta(\Gamma, \mathcal{A}) = \{\alpha_1, \ldots, \alpha_\ell\} \in H^2(BT^n)$ such that $\varphi(\alpha_i) = \tau_i - \tau_{i+1}$ for $i = 1, \ldots, \ell$, where τ_i is the Thom class of the $(n-1)$-valent torus subgraph Γ_i. Then, one of the following cases occur:*

The 1^{st} case: *if $\tau_1 \cdots \tau_{\ell+1} = 0$ and $\cap_{i\in I}\tau_i \neq 0$ for all $I \subset [\ell+1]$ with $|I| = \ell$, i.e., $\Gamma_1 \cap \cdots \cap \Gamma_{\ell+1} = \emptyset$ but $\cap_{i\in I}\Gamma_i \neq \emptyset$, then there is the fibration*

$$\rho : (\Gamma, \mathcal{A}) \to (K_{\ell+1}, \mathcal{A}_{\ell+1});$$

The 2^{nd} case: *otherwise, i.e., $\Gamma_1 \cap \cdots \cap \Gamma_{\ell+1} \neq \emptyset$, there is the blow-up $(\widetilde{\Gamma}, \widetilde{\mathcal{A}}) \to (\Gamma, \mathcal{A})$ along $\Gamma' = \Gamma_1 \cap \cdots \cap \Gamma_{\ell+1}$ such that $(\widetilde{\Gamma}, \widetilde{\mathcal{A}})$ satisfies the 1^{st} case.*

In the statement of theorem, $\mathcal{A}_{\ell+1}$ is the standard axial function of the complete graph $K_{\ell+1}$ which defined by $\mathcal{A}_{\ell+1}(p_0p_j) = \alpha_j$ and $\mathcal{A}_{\ell+1}(p_ip_j) = \alpha_j - \alpha_i$ for $i, j \neq 0$. Namely, $(K_{\ell+1}, \mathcal{A}_{\ell+1})$ is the torus graph which is obtained by the standard T^n-action on $\mathbb{C}P^n$.

Remark 3.6. Note that in [**Ku**] we announced an analogues result for all GKM graphs with root systems of type A. However, in general, the GKM blow-up is not well-defined for GKM graphs. So we need to change the statement of the main theorem in [**Ku**] as above.

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