A Carleman estimate for an elliptic operator with a discontinuous coefficient in a partially anisotropic media

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Abstract

We prove a Carleman estimate for the elliptic operator $A = -\nabla \cdot (B\nabla)$ with an arbitrary observation region. The structure of the $n\times n$ matrix B gives the partially anisotropic character: a block diagonal matrix in which the first block is the product of a $(n-1)\times(n-1)$ hermitian matrix C_τ by a scalar function $x_n \to a(x_n)$ and the second block c is a positive function. The coefficients of the matrix C_τ are $\mathscr{C}^1$ and a, c are piecewise $\mathscr{C}^1$ in $\overline{\Omega}$, a bounded connected domain of $\mathbb{R}^n$. If S denotes the set where discontinuities of c can occur, we suppose that Ω is stratified in a neighborhood of S in the sense that locally it takes the form $\Omega_\delta := \Omega' \times (-\delta, \delta)$ with $\Omega' \subset \mathbb{R}^{n-1}$, $\delta > 0$ and $S = \Omega' \times \{0\}$. This Carleman estimate is obtained through the introduction of a suitable mesh of Ω_δ and an associated approximation of B involving the Carleman large parameters.
We shall give some extensions of the used method and explain how a local estimate makes us able of writing a global estimate.

1 Notations and result

We consider the positive selfadjoint elliptic operator: $D(A) = \{u \in H_0^1(\Omega); \nabla\cdot(B\nabla u) \in L^2(\Omega)\}, A = -\nabla \cdot B\nabla$, where for the simplicity we assume Ω is the bounded open set $\Omega = \Omega' \times (-H, H) \subset \mathbb{R}^{n-1} \times \mathbb{R}$. We suppose

- the interface $S := \{x = (x', 0); x' \in \Omega'\}$ is a $\mathscr{C}^2$ open set;
- the $n \times n$ symmetric real matrix B has the form

$$B(x) = \begin{pmatrix} a(x_n)C_\tau(x) & 0 \\ 0 & c(x) \end{pmatrix}; \tag{1}$$

- the function $x_n \to a(x_n)$ is $\mathscr{C}^1$ on $[-H, 0]$ and $[0, H]$, the scalar function $x \to c(x)$ is $\mathscr{C}^1$ on the closure of the two open sets $\Omega^\pm = \{x \in \Omega; \pm x_n > 0\}$, its two restrictions $x' \to c_\pm(x') := c(x', 0^\pm)$ to the interface S being $\mathscr{C}^2$, whereas the coefficients of the $(n-1)\times(n-1)$ matrix C_τ belong to $C^1(\bar{\Omega})$;
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$$\begin{aligned} 0 < c_{\min} \le a(x_n), c(x) \le c_{\max} < \infty, x \in \Omega \\ 0 < c_{\min} Id_{n-1} \le C_\tau \le c_{\max} Id_{n-1} < \infty, x \in \Omega; \end{aligned} \tag{2}$$

- ω is a fixed open set with $\omega \Subset \Omega^+$;
- $\delta, 0 < \delta < H$, is a real number small enough in order that $\Omega' \times (0, \delta)$ does not contains entirely ω;
- φ is a weight function such that $\varphi(x) = e^{\lambda\beta(x)}, \beta \in \mathscr{C}^0(\bar{\Omega}), \lambda > 0$.

Note that the functions a **and** c **can be discontinuous on** $x_n = 0$. Our main result is the Carleman estimate given by

Theorem 1. *There exist three strictly positive constants* C, λ_0, s_0 *and a function* $\varphi, \varphi(x) = e^{\lambda\beta(x)}$, *with* $\beta \in C^0(\bar{\Omega})$, *such that*

$$s\lambda^2 \|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + s^3\lambda^4 \|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)} + s\lambda\left(|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla_\tau u|^2_{L^2(S)} + |e^{s\varphi}\varphi^{\frac{1}{2}}\partial_{x_n}u_{|S^\pm}|^2_{L^2(S)}\right) + s^3\lambda^3 |e^{s\varphi}\varphi^{\frac{3}{2}}u_{|S}|^2_{L^2(S)} \leq C\left(\|e^{s\varphi}Au\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}\right) \quad (3)$$

for all $u \in D(A), \lambda \geq \lambda_0$ *and* $s \geq s_0$.

2 Some ideas on the used approach

We shall begin to prove the result in a local case, i.e. in $\Omega_\delta := \Omega' \times (-\delta, \delta), 0 < \delta < H$, where $\delta, 0 < \delta < H$, is a real number small enough in order that $\Omega' \times (0, \delta)$ does not contain entirely ω. In this configuration, the last term of (3) does nos exist.
Then, the global estimate (3) in Ω is a consequence of works of other authors, by example [5]. We shall give indications.

We start from the following estimate which is wellknown by the specialists:
There exist on Ω_δ *a weight function* $x_n \to \beta(x_n)$, *non decreasing since we assume* $\omega \subset \Omega^+$, *and four strictly positive constants* C, C', λ_0, s_0 *such that*

$$C\Big(s\lambda^2\|\varphi^{\frac{1}{2}}e^{s\varphi}\nabla u\|^2_{L^2(\Omega_\delta)} + s^3\lambda^4\|\varphi^{\frac{3}{2}}e^{s\varphi}u\|^2_{L^2(\Omega_\delta)}\Big) + s\lambda\varphi_{|S}\left(\int_S [c^2\beta' |e^{s\varphi}\partial_{x_n}u|^2]_S\, d\sigma + \int_S |s\lambda\varphi e^{s\varphi}u_{|S}|^2 [c^2\beta'^3]_S\, d\sigma\right) \leq C'\|e^{s\varphi}Au\|^2_{L^2(\Omega_\delta)} + s\lambda\varphi_{|S}\int_S |e^{s\varphi}\nabla_\tau u|^2 \|[\beta' caC_\tau]_S\|\, d\sigma \quad (4)$$

$\forall u \in D(A), \lambda \geq \lambda_0, s \geq s_0$ *and* $\operatorname{supp} u \subset \overline{\Omega'} \times (-\delta, \delta)$.
In (4), ∇_τ is the gradient relatively to the horizontal coordinates $x' = (x_1, \ldots, x_{n-1})$ and $[f]_S(x') := f(x', 0^+) - f(x', 0^-)$ is the jump of the function (resp. matrix) f across the interface S in $L^\infty(S)$.
So, it is clear that we have to estimate $s\lambda\varphi_{|S}\int_S |e^{s\varphi}\nabla_\tau u|^2 \|[\beta' caC_\tau]_S\|\, d\sigma$ in order to prove that the left hand side can absorb it.

In a first step, we assume that the function $x_n \to a(x_n)$ is equal to 1. The introduction of a partition of unity on the interface allows us to recover Ω_δ by small cubes, indexed by (j, ℓ), in which we approach the tangential operators $A_\tau := -\nabla_\tau \cdot C_\tau \nabla_\tau$ by operators with constant coefficients in each cube. So, from $Au = f$ we deduce the sequence of problems

$$\left\|\begin{array}{l} -\nabla_\tau \cdot (C_\tau^{j,\ell}\nabla_\tau u_{j,\ell}) - c^{j,\ell}\partial^2_{x_n}u_{j,\ell} = f_{j,\ell} + \tilde{g}_{j,\ell} + \tilde{h}_{j,\ell} + t_{j,\ell}, \\ \qquad\qquad \text{avec} \\ \tilde{g}_{j,\ell} := (c - c^{j,\ell})\partial^2_{x_n}u_{j,\ell}, \\ \tilde{h}_{j,\ell} := [A, \Upsilon_\ell \chi_j]u + (\partial_{x_n}c)\partial_{x_n}u_{j,\ell}, \\ t_{j,\ell} := \nabla_\tau \cdot ((C_\tau - C_\tau^{j,\ell})\nabla_\tau u_{j,\ell}), \end{array}\right. \quad (5)$$

with the following transmission conditions

$$[u_{j,\ell}]_S = 0, [c^{j,\ell}\partial_{x_n} u_{j,\ell}]_S = [(c^{j,\ell} - c)\partial_{x_n} u_{j,\ell}]_S := \theta_{j,\ell}. \tag{6}$$

By projection on a complete system of eigenfunctions of A_τ we arrive to a sequence of ordinary differential equations

$$\begin{array}{c} \mu^2_{j,\ell,k} u_{j,\ell,k} - c^{j,\ell}\partial^2_{x_n} u_{j,\ell,k} = f_{j,\ell,k} + \tilde{g}_{j,\ell,k} + \tilde{h}_{j,\ell,k} + t_{j,\ell,k} \\ [u_{j,\ell,k}]_S = 0, [c^{j,\ell}\partial_{x_n} u_{j,\ell,k}]_S = \theta_{j,\ell,k}. \end{array} \tag{7}$$

Two points are crucial: the size of these cubes and the treatment of the operator A_τ in the cubes included in a neighborhood of the boundary and the interface at once.

The second step concerns the case without the hypothesis $a(x_n) = 1$.

3 Some comments

According to remaining time we shall compare several recent works ([4], [5],[6], [2]). Each time the trace of the solution $ue^{s\varphi}$ on the interface is evaluated, and we shall see the limits and advantages of cited works. In our case we evaluate u multiplied by a constant: the value of the trace of φ on S. Our method ([1]) allows us to obtain Therorem 1 when the interface S and $\partial\Omega$ are transverse. Many questions are open, in particular the case of a parabolic operator with discontinuous coefficients (the first work is [3]).

References

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