Increasing stability in the inverse source and conductivity problems.

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1 Increasing stability for sources

Let

$$(\Delta + k^2)u = -f_1 - ikf_0 \text{ in } \mathbb{R}^3, \tag{1.1}$$

with the Sommerfeld radiation condition. $f_0, f_1 \in L^2(\Omega)$ and are zero outside a bounded domain Ω with $\partial\Omega \in C^1$.

We are interested in uniqueness and stability of f_0, f_1 from the near field data

$$u = u_0,\ \partial_\nu u = u_1 \ \text{ on } \ \Gamma, \text{when } 0 < k < K, \tag{1.2}$$

where Γ is an open part of $\partial\Omega$. For any non empty Γ and any K one can show uniqueness of f_0, f_1. For better than logarithmic stability we assume that $\Gamma = \partial\Omega$.

Theorem 1.1. *[1]*
There is $C = C(\Omega)$ such that

$$\|f_0\|^2_{(1)}(\Omega) + \|f_1\|^2_{(0)}(\Omega) \leq C(\varepsilon^2 + M_2^2 \frac{1}{1 + (K)^{\frac{4}{3}} E^{\frac{1}{2}}})$$

for all $u \in H_{(2)}(\Omega)$ solving (1.1), (1.2), where

$$\varepsilon^2 = \int_0^K ((1 + k^2)\|u(,k)\|^2_{(0)}(\partial\Omega) + \|\nabla u(,k)\|^2_{(0)}(\partial\Omega))dk,\ E = -ln\varepsilon$$

and $\|f_0\|_{(2)}(\Omega) + \|f_1\|_{(1)}(\Omega) \leq M_2$.

In [1] there are proofs based on sharp bounds of special harmonic measure to trace dependence on K of analytic continuation of $u(,k)$ from $(0, K)$ onto all real k, the Fourier transform in k into time domain, and exact boundary observability for the corresponding wave equation. We also give a strong numerical evidence of increasing resolution for larger K.

2 Increasing stability for conductivity coefficient

The stationary electromagnetic field $(\mathbf{E}, \mathbf{H})$ of frequency k satisfies

$$curl\mathbf{E} = ik\mathbf{H} \text{ in } \Omega,$$

$$curl\mathbf{H} = \sigma\mathbf{E} - ik\mathbf{E}, \tag{2.3}$$

where magnetic and electric permeabilities are assumed to be 1, and σ is electric conductivity of a bounded C^1 three-dimensional domain Ω.

We are interested in better stability of revovery of σ from the complete Cauchy data

$$\mathbf{C} = \{(\nu \times \mathbf{E}, \nu \times \mathbf{H})\} \text{ on } \ \partial\Omega.$$

Let

$$\varepsilon = sup\ sup\ inf \frac{||(\nu \times \mathbf{E}(j), \nu \times \mathbf{H}(j)) - (\nu \times \mathbf{E}(l), \nu \times \mathbf{H}(l))||_{TH(\partial\Omega)}}{||\nu \times \mathbf{E}(j)||_{TH(\partial\Omega)}},$$

where inf is over tangential traces of $\mathbf{E}(l), \mathbf{H}(l)$, next sup is over $\mathbf{E}(j), \mathbf{H}(j)$ and outer sup is over $j \neq l$, $\mathbf{E}(j), \mathbf{H}(j)$ solve (2.3) with $\sigma = \sigma(j), j = 1, 2$, the norms are in known tangential traces of solutions with $curl$ in $L^2(\Omega$, and $\mathcal{E} = -ln\epsilon$.

Theorem 2.1. *[3]*
Suppose that $supp(\sigma(1) - \sigma(2)) \subset \Omega$.
There are $C = C(\Omega), m = m(\Omega)$ *such that if* $1 < k$ *and*

$$||\sigma_j||_{(4)}(\Omega) < m,$$

then

$$||\sigma_1 - \sigma_2||_{(-2)}(\Omega) \leq \frac{C}{k + \mathcal{E}} + \frac{C(k + \mathcal{E})^3}{k} \epsilon^{\frac{1}{2}}.$$

Proofs in [3] are based on a reduction to a vectorial Schrödinger equation and use of complex and real geometrical optics as initiated in [2].

Both Theorems suggest that logarithmic (unstable) component in stability bounds is decreasing when k grows.

References

[1] Jin Cheng, Victor Isakov, Shuai Lu, Increasing stability in the inverse source problem with many frequencies, *J. Diff. Equat.*, **260** (2016), 4786-4804.

[2] Victor Isakov, Increasing stability for the Schrödinger potential from the Dirichlet-to Neumann map, *Discr. Cont. Dyn. Syst. S*, **4** (2011) 631-641.

[3] Victor Isakov, Ru-Yu Lai, Jenn-Nan Wang, Increasing stability for the conductivity and attenuation coefficients, *SIAM J. Math. Anal.*, (2016) (to appear).