Increasing stability in the inverse source and conductivity problems.

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1 Increasing stability for sources

Let
\[ (\Delta + k^2)u = -f_1 - ikf_0 \text{ in } \mathbb{R}^3, \] with the Sommerfeld radiation condition. \( f_0, f_1 \in L^2(\Omega) \) and are zero outside a bounded domain \( \Omega \) with \( \partial \Omega \in C^1 \).

We are interested in uniqueness and stability of \( f_0, f_1 \) from the near field data
\[ u = u_0, \partial_\nu u = u_1 \text{ on } \Gamma, \text{when } 0 < k < K, \] (1.2)
where \( \Gamma \) is an open part of \( \partial \Omega \). For any non empty \( \Gamma \) and any \( K \) one can show uniqueness of \( f_0, f_1 \). For better than logarithmic stability we assume that \( \Gamma = \partial \Omega \).

Theorem 1.1. [1]
There is \( C = C(\Omega) \) such that
\[ \|f_0\|_{(1)}^2(\Omega) + \|f_1\|_{(0)}^2(\Omega) \leq C(\varepsilon^2 + M_2^2 \frac{1}{1 + (K)^4 E^2}) \]
for all \( u \in H(2)(\Omega) \) solving (1.1), (1.2), where
\[ \varepsilon^2 = \int_0^K ((1 + k^2)\|u(\cdot, k)\|^2_{(0)}(\partial \Omega) + \|\nabla u(\cdot, k)\|^2_{(0)}(\partial \Omega))dk, \quad E = -ln\varepsilon \]
and \( \|f_0\|_{(2)}(\Omega) + \|f_1\|_{(1)}(\Omega) \leq M_2 \).

In [1] there are proofs based on sharp bounds of special harmonic measure to trace dependence on \( K \) of analytic continuation of \( u(\cdot, k) \) from \((0, K)\) onto all real \( k \), the Fourier transform in \( k \) into time domain, and exact boundary observability for the corresponding wave equation. We also give a strong numerical evidence of increasing resolution for larger \( K \).

2 Increasing stability for conductivity coefficient

The stationary electromagnetic field \((E, H)\) of frequency \( k \) satisfies
\[ \text{curl} E = ikH \text{ in } \Omega, \]
\[ \text{curl} H = \sigma E - ikE, \] (2.3)
where magnetic and electric permeabilities are assumed to be 1, and \( \sigma \) is electric conductivity of a bounded \( C^1 \) three-dimensional domain \( \Omega \).

We are interested in better stability of recovery of \( \sigma \) from the complete Cauchy data
\[ C = \{(\nu \times E, \nu \times H)\} \text{ on } \partial \Omega. \]
Let
\[ \varepsilon = \sup \sup \inf \frac{||\nu \times \mathbf{E}(j), \nu \times \mathbf{H}(j)) - (\nu \times \mathbf{E}(l), \nu \times \mathbf{H}(l))||_{TH(\partial \Omega)}}{||\nu \times \mathbf{E}(j)||_{TH(\partial \Omega)}} \]
where \(\inf\) is over tangential traces of \(\mathbf{E}(l), \mathbf{H}(l)\), next \(\sup\) is over \(\mathbf{E}(j), \mathbf{H}(j)\) and outer \(\sup\) is over \(j \neq l\), \(\mathbf{E}(j), \mathbf{H}(j)\) solve (2.3) with \(\sigma = \sigma(j), j = 1, 2\), the norms are in known tangential traces of solutions with \(\text{curl}\) in \(L^2(\Omega)\), and \(\mathcal{E} = -\ln \varepsilon\).

**Theorem 2.1.** [3]

Suppose that \(\text{supp}(\sigma(1) - \sigma(2)) \subset \Omega\).

There are \(C = C(\Omega), m = m(\Omega)\) such that if \(1 < k\) and
\[ ||\sigma_j||_{(4)}(\Omega) < m, \]
then
\[ ||\sigma_1 - \sigma_2||_{(-2)}(\Omega) \leq \frac{C}{k + \mathcal{E}} + \frac{C(k + \mathcal{E})}{k} \varepsilon^2. \]

Proofs in [3] are based on a reduction to a vectorial Schrödinger equation and use of complex and real geometrical optics as initiated in [2].

Both Theorems suggest that logarithmic (unstable) component in stability bounds is decreasing when \(k\) grows.

**References**

