Inversion at the boundary for dynamical elastic equation

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Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$. Let us consider the following initial boundary value problem for system of equations for linear elasticity

$$\begin{cases} \rho\partial_t^2 u = \mathrm{div}(\mathbb{C}\varepsilon(u)) \ \text{ in } \Omega \times (0,T) \\ u = f = O(t^2) \ \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = \partial_t u(x,0) = 0 \ \text{ in } \Omega. \end{cases}$$

Here u denotes the displacement vector and $\varepsilon(u) = (\varepsilon_{ij}(u)) = (\nabla u + (\nabla u)^T)/2$ the strain tensor. Furthermore, $\mathbb{C} = \mathbb{C}(x) = (C_{ijkl}(x))$ is the elasticity tensor and ρ is the density. We assume that $\mathbb{C}$ is isotropic, i.e.,

$$C_{ijkl}(x) = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with Kronecker's delta δ_{ij} and Lamé moduli λ, $\mu \in C^\infty(\overline{\Omega})$ such that $\mu > 0$ and $\lambda + 2\mu > 0$ on $\overline{\Omega}$.

We define the dynamical Dirichlet to Neumann map (DN map) Λ_T by

$$\Lambda_T : H^2((0,T); H^{3/2}(\partial\Omega)) \ni f = O(t^2) \mapsto (\mathbb{C}\varepsilon(u))\nu|_{\partial\Omega} \in W((0,T); \partial\Omega),$$

where u solves the above initial boundary value problem and $\mathbb{C}\varepsilon(u)$ is a 3×3 matrix with its (i,j) component $(\mathbb{C}\varepsilon(u))_{ij}$ given by $(\mathbb{C}\varepsilon(u))_{ij} = \sum_{i,j=1}^3 C_{ijkl}\varepsilon_{kl}(u)$. Here we defined $W((0,T); \partial\Omega) := \cap_{j=1}^2 H^{3-j}((0,T); H^{j-3/2}(\partial\Omega))$. In this paper, we consider the inverse problem of recovering λ, μ, ρ, as well as all their derivatives at the boundary $\partial\Omega$ from the full symbol of DN map Λ_T.

Theorem *There is an inversion formula for identifying λ, μ, ρ and all their derivatives on $\partial\Omega$ from the Λ_T.*

The uniqueness of identifying λ, μ, ρ at the boundary was shown in [5], but since then giving a formula to recover them has been left open for 15 years. One of the reasons for this is that there are two metrics involved in the dynamical isotropic system of equations which are very hard to separate at the boundary using some special solutions such as high frequency asymptotic solutions or progressive wave solutions associated to interior P wave

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and S wave with respective speeds $\sqrt{(\lambda+2\mu)/\rho}$ and $\sqrt{\mu/\rho}$, because they are coupled at the boundary.

The key element of the proof of this theorem is that we can get the asymptotic formula of the DN map Λ for some elliptic system of equations with a large parameter τ obtained from Λ_T via a finite time Laplace transform. Then we can identify the parameters from Λ. We note that the parameter τ is nothing but the Laplace variable of this transform.

The method we used here for the elliptic system has been used successfully for static inverse problems for a long time. (See [2], [3], [4], [7].)

Knowing all the derivatives of the coefficients at the boundary, we can actually get an approximation of DN map $\Lambda_T(s)$ on the boundary $\Gamma(s)$ of $\Omega(s) = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > s\}$ for $0 < s \ll 1$. Then, we could get an approximation of the coefficients at $\Gamma(s)$. Repeating this process, we can obtain an approximation for λ, μ, ρ in the interior, layer by layer, from the DN map Λ_T. This algorithm is called layer stripping. It was first developed for the electrical impedance tomography problem (EIT) in [1], [6]. Nakamura, Tanuma and Uhlmann [4] gave a layer stripping algorithm for transversely isotropic elasticity in static case.

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