Inverse problems for fractional differential equations: some overview

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and A be a regular elliptic operator of the second order.

We consider

$$\begin{cases} D_t^\mu u(x,t) = Au(x,t) + F(x,t), \quad x \in \Omega,\, t > 0, \\ u(x,0) = a(x), \quad x \in \Omega, \\ u|_{\partial\Omega\times(0,\infty)} = g. \end{cases} \tag{1}$$

Here

$$\partial_t^\xi \psi(t) = \frac{1}{\Gamma(1-\xi)} \int_0^t (t-s)^{-\xi} \frac{d\psi}{ds}(s)ds, \quad \xi \in (0,1)$$

is the Caputo fractional derivative and we set

$$D_t^\mu \psi(t) = \int_0^1 \mu(\xi) \partial_t^\xi \psi(t) d\xi.$$

Example 1: $\mu(\xi) = \delta_\alpha$ with fixed $\alpha \in (0,1)$. Then $D_t^\mu \psi(t) = \partial_t^\alpha \psi(t)$ and (1) describes an initial-boundary value problem for a fractional diffusion equation.

Example 2: $\mu(\xi) = \sum_{i=1}^{\ell} q_i \delta_{\alpha_i}$ where $q_i(x,t) > 0$ and

$$0 < \alpha_1 < \cdots < \alpha_\ell < 1.$$

Then (1) describes an initial-boundary value problem for a multi-term time fractional diffusion equation.

I present recent results by me and my colleagues on the direct problems and various inverse problems. For the direct problem, I intend to construct a comprehensive theory and here I describe some important starting steps.

Grand Research Plan