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FAST SUBSPACE OPTIMIZATION METHOD FOR NONLINEAR INVERSE PROBLEMS IN BANACH SPACES WITH UNIFORMLY CONVEX PENALTY TERMS

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In this talk, we aim to solve the inverse problem of the following form

$$F(x) = y, \tag{1}$$

where $F : \mathcal{D}(F) \subset \mathcal{X} \to \mathcal{Y}$ is a nonlinear operator between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. We are in particular interested in the case that we only have noisy data y^δ satisfying

$$\left\| y^\delta - y \right\| \le \delta,$$

where the noise level $\delta > 0$ is known. Due to the inherent ill-posedness of inverse problems, some regularization methods should be used to produce a stable approximate solution of (1).

Landweber iteration is one of the most prominent regularization methods for solving nonlinear inverse problems due to its simplicity, see [1] and reference therein. In order to capture the special features of the sought solutions, such as sparsity and discontinuities, the penalty term is allowed to be non-smooth to include L^1 and total variation (TV) like penalty functionals. Let $\theta : \mathcal{X} \to (-\infty, \infty]$ be a proper, lower semi-continuous, convex function, then the method in [1] has the form of

$$\xi_{n+1}^\delta = \xi_n^\delta - t_n^\delta F'\left(x_n^\delta\right)^* J_r^{\mathcal{Y}} \left(F\left(x_n^\delta\right) - y^\delta\right), \quad x_{n+1}^\delta = \arg\min_{x \in \mathcal{X}} \left\{\theta\left(x\right) - \left\langle \xi_{n+1}^\delta, x\right\rangle\right\} \tag{2}$$

with the suitable chosen step size t_n^δ, $J_r^{\mathcal{Y}}$ denotes the duality mapping of $\mathcal{Y}$ with the gauge function $t \to t^{r-1}$. This method could give satisfactory reconstructed solutions, however, the slow convergence of (2) makes it inefficient in practical applications. Hence, it is necessary to introduce some accelerate strategies to fasten its convergence speed.

One starting point is to use the sequential subspace optimization (SESOP) method, which was first proposed to solve the large scale well-posed optimization problems by utilizing multiple search directions per iteration. Then it was extended to solve the linear inverse problems [2,3] and nonlinear inverse problems [4,5]. The SESOP method in [5], however, only covers $\theta\left(x\right) = \|x\|^s/s$ with $1 < s < \infty$ on uniformly convex and uniformly smooth Banach spaces and thus excludes the use of the L^1 and TV like penalty functionals. Also the duality mapping $J_r^{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}^*$ requires $r = 2$ to guarantee the convergence and regularization of the method in [5], which is not suitable for the data containing non-Gaussian noise.

Aiming at accelerating the iteration (2), we formulate an extension of the SESOP method in the sprit of [5] to solve nonlinear inverse problems with both $\mathcal{X}$ and $\mathcal{Y}$ being Banach spaces and the non-smooth convex function $\theta : \mathcal{X} \to (-\infty, \infty]$ is incorporated as the penalty term so that the method can be used for sparsity reconstruction and discontinuity detection. Then the proposed SESOP method with convex penalty can be formulated as

$$\xi_{n+1}^\delta = \xi_n^\delta - \sum_{i \in I_n} t_{n,i}^\delta F'\left(x_i^\delta\right)^* J_r^{\mathcal{Y}} \left(F\left(x_i^\delta\right) - y^\delta\right), \quad x_{n+1}^\delta = \arg\min_{x \in \mathcal{X}} \left\{\theta\left(x\right) - \left\langle \xi_{n+1}^\delta, x\right\rangle\right\}, \tag{3}$$

where $J_r^{\mathcal{Y}}$ denotes the duality mapping of $\mathcal{Y}$ with $1 < r < \infty$ and $I_n \subseteq \{n - N + 1, \cdots, n\}$ is a finite index set with given $N > 1$. A detailed convergence analysis and regularization results are given when the proposed method is terminated by the discrepancy principle. Finally, we present some numerical examples for parameter identification problems to validate the theoretical analysis and to verify the efficiency of the proposed method.

One-dimensional Sparsity Reconstruction. In this subsection, we assume that the sought parameter is sparse and the data only contains Gaussian noise with $\delta = 0.5\%,\ 0.1\%,\ 0.05\%$. For this situation, we take $\theta\left(x\right) = \frac{1}{2\beta}\int_\Omega \left|x\left(w\right)\right|^2 dw + \int_\Omega \left|x\left(w\right)\right| dw$ with $\beta > 0$.

The numerical results by our proposed SESOP method with the number M of search directions, called SESOP-M in the following, and Landweber iteration in [1] are summarized in Table 1. Observe that the SESOP-2 method could produce the same accurate solutions but requires much less iteration numbers and computation time, validating its acceleration effect.

Table 1: Numerical results of one-dimensional sparsity reconstruction for noisy data.

	$\delta = 0.5\%$			$\delta = 0.1\%$			$\delta = 0.05\%$		
Method	n_δ	Time(s)	RE	n_δ	Time(s)	RE	n_δ	Time(s)	RE
Landweber	2698	10.5514	0.8044	37721	142.1369	0.2553	47036	193.6245	0.1191
SESOP-1	686	3.4247	0.8089	10271	49.0497	0.2542	17861	86.4013	0.1192
SESOP-2	212	1.7004	0.8073	4091	27.8389	0.2553	6614	47.5240	0.1188

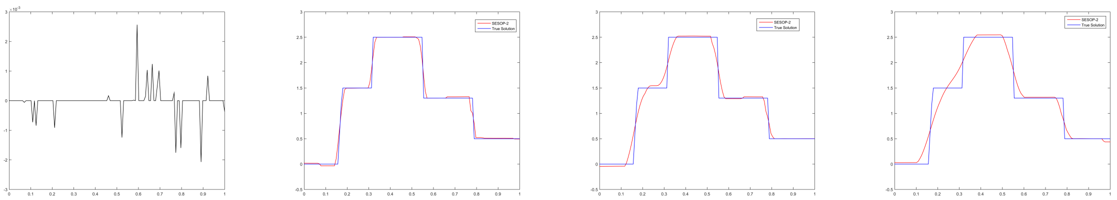
One-dimensional Discontinuity Detection. In this subsection, we consider that the sought parameter is piecewise constant. For this case, we choose $\theta(x) = \frac{1}{2\beta}\int_\Omega |x(w)|^2 dw + TV(x)$, where $\beta > 0$ and $TV(x)$ denotes the total variation of x.

We first consider that the data only contains Gaussian noise with various noise levels. The numerical results by SESOP-2, SESOP-1 and Landweber iteration are displayed in Table 2. It is shown that SESOP-2 could lead to a significant reduction of iteration numbers through comparison with Landweber type iteration.

Table 2: Numerical results of one-dimensional discontinuity detection under Gaussian noise.

	$\delta = 0.5\%$			$\delta = 0.1\%$			$\delta = 0.05\%$		
Method	n_δ	Time(s)	RE	n_δ	Time(s)	RE	n_δ	Time(s)	RE
Landweber	2973	18.0963	0.1181	27518	207.8141	0.0696	37927	308.1425	0.0653
SESOP-1	1242	14.6804	0.1181	17087	145.4954	0.0697	26952	214.6641	0.0653
SESOP-2	600	13.1969	0.1182	8430	94.7757	0.0695	11862	130.0063	0.0653

We also investigate the effect of Banach spaces when the data contains impulsive noise. The impulsive noise is plotted in Figure 1(a). Figure 1(b)-(d) present the reconstructions by SESOP-2 using Banach space $\mathcal{Y} = L^r[0,1]$ with $r = 2$, $r = 1.5$ and $r = 1.1$, respectively. Observe that $L^{1.1}$ misfit data terms could produce a more accurate solution, which verifies that the L^r misfit data terms with $r > 1$ close to 1 are suitable for this kind of noise.



(a) (b) (c) (d)

Figure 1: The one-dimensional discontinuity detection under Impulsive noise. (a) Impulsive noise; (b) Reconstruction by SESOP-2 with $r = 1.1$; (c) Reconstruction by SESOP-2 with $r = 1.5$; (d) Reconstruction by SESOP-2 with $r = 2$.

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