# Direct and inverse scattering problems for the local perturbation of an open periodic waveguide in the half plane

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Let $k > 0$ be the wave number, and let $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in $\mathbb{R}^2_+$. We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a > 0$. Let $n \in L^\infty(\mathbb{R}^2_+)$ be real value, $2\pi$-periodic with respect to $x_1$ (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_0 > 0$ such that $n \geq n_0$ in $\mathbb{R}^2_+$. Let $q \in L^\infty(\mathbb{R}^2_+)$ be real value with the compact support in $W$. We denote by $Q := \mathrm{supp} q$.

We consider the following scattering problem: For fixed $y \in \mathbb{R}^2_+ \setminus \overline{W}$, determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u^s + k^2(1+q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{0.1}$$

$$u^s = 0 \text{ on } \Gamma_0, \tag{0.2}$$

Here, the incident field $u^i$ is given by $u^i(x, y) = G_n(x, y)$, where $G_n$ is the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + k^2 n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \tag{0.3}$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function in $\mathbb{R}^2_+$ for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of $y$ at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi_k(x, y) := \frac{i}{4} H^{(1)}_0(k|x - y|), \;\; x \neq y. \tag{0.4}$$

$\tilde{u}^s$ is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, $\tilde{u}^s$ vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2 n \tilde{u}^s = k^2(1 - n)G(\cdot, y) \text{ in } \mathbb{R}^2_+. \tag{0.5}$$

If we impose a suitable radiation condition introduced by Kirsch and Lechleiter (see Definition 6.7 in [4]), the unperturbed solution $\tilde{u}^s$ is uniquely determined.

In order to show the well-posedness of the perturbed scattering problem (0.1)–(0.2), we make the following assumption.

**Assumption 0.1.** *We assume that $k^2$ is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H^1_0(\mathbb{R}^2_+)$, that is, evey $v \in H^1(\mathbb{R}^2_+)$ which satisfies*

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}^2_+, \tag{0.6}$$

$$v = 0 \text{ on } \Gamma_0, \tag{0.7}$$

*has to vanish for $x_2 > 0$.*

Our first aim is to show the following theorem under Assumption 0.1.

**Theorem 0.2** (Theorem 1.2 in [1]). *Let Assumptions 0.1 hold and let* $f \in L^2(\mathbb{R}^2_+)$ *such that* $\mathrm{supp} f = Q$. *Then, there exists a unique solution* $u \in H^1_{loc}(\mathbb{R}^2_+)$ *such that*

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}^2_+, \tag{0.8}$$

$$u = 0 \text{ on } \Gamma_0, \tag{0.9}$$

*and* $u$ *satisfies the radiation condition in the sense of Definition 6.7 in [4].*

Roughly speaking, this radiation condition requires that we have a decomposition of the solution $u$ into $u^{(1)}$ which decays in the direction of $x_1$, and a finite combination $u^{(2)}$ of *propagative modes* which does not decay in $x_1$, but it exponentially decays in $x_2$.

By the well-posedness of this perturbed scattering problem, we are now able to consider the inverse problem of determing the supprot of $q$ from measured scattered field $u^s$ by the incident field $u^i$. Let $M := \{(x_1, m) : a < x_1 < b\}$ for $a < b$ and $m > h$, and $Q := \mathrm{supp} q$. With the scattered field $u^s$, we define the *near field operator* $N : L^2(M) \to L^2(M)$ by

$$Ng(x) := \int_M u^s(x,y)g(y)ds(y), \ x \in M. \tag{0.10}$$

The inverse problem we consider here is to determine support $Q$ of $q$ from the scattered field $u^s(x,y)$ for all $x$ and $y$ in $M$ with one $k > 0$. In other words, given the near field operator $N$, determine $Q$.

Our second aim is to provide the following theorem based on the idea of the *monotonicity mathod* (e.g., see [3]).

**Theorem 0.3** (Theorem 1.1 in [2]). *Let* $B \subset \mathbb{R}^2$ *be a bounded open set. Let Assumption hold, and assume that there exists* $q_{min} > 0$ *such that* $q \geq q_{min}$ *a.e. in* $Q$. *Then for* $0 < \alpha < k^2 n_{min} q_{min}$,

$$B \subset Q \quad \Longleftrightarrow \quad \alpha H^*_B H_B \leq_{\text{fin}} \mathrm{Re} N, \tag{0.11}$$

*where the operator* $H_B : L^2(M) \to L^2(B)$ *is given by*

$$H_B g(x) := \int_M \overline{G_n(x,y)} g(y)ds(y), \ x \in B, \tag{0.12}$$

*and the inequality on the right hand side in (0.11) denotes that* $\mathrm{Re}N - \alpha H^*_B H_B$ *has only finitely many negative eigenvalues, and the real part of an operator* $A$ *is self-adjoint operators given by* $\mathrm{Re}(A) := \frac{1}{2}(A + A^*)$.

By Theorem 0.3, we understand whether an artificial domain $B$ is contained in $Q$ or not. Then, by dispersing a lot of balls $B$ in $\mathbb{R}^2_+$ and for each $B$ checking (0.11) we can reconstruct the shape and location of unknown $Q$.

## References


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