# Numerical Realizations of X-ray Computerized Tomography by Cauchy-type Boundary Integration

Hiroshi Fujiwara (Kyoto University)
Alexandru Tamasan (University of Central Florida)

We study numerical feasibility and properties of a Cauchy-type boundary integration formula in X-ray Computerized Tomography (CT). The integration formula has been developed by Bukhgeim et. al. [1, 2, 3] based on mainly the theory of $A$-analytic functions by converting the mathematical model of X-ray CT into the inverse source problem for the transport equation. However, to the best of the authors' knowledge, its numerical realization have not been realized so far. It is also worthy of notice that the method has a possibility in application to the radiative transport equation [4], while the inverse Radon transform is unavailable due to its crucial dependence on straightness of X-ray propagation.

Let $D$ be a convex domain in $\mathbb{R}^2$, and $\mu(x)$ be the absorption coefficient at $x \in D$. The standard mathematical model of the X-ray CT [7] is to find $\mu(x)$ satisfying the integral equation

$$\int_{x \cdot \omega = s} \mu(x)\, d\ell = R\mu(\omega, s),$$

from measurement of $R\mu(\omega, s)$ for all $\omega \in S^1$ and $s \in \mathbb{R}$, which is called the Radon transform of $\mu$. The filtered back projection (FBP) with the inverse Radon transform [8] has been well established in reconstruction of $\mu(x)$, which is given as

$$\mu(x) \approx \frac{1}{2\pi} \int_0^{\pi} P_\omega * h(x \cdot \omega)\, d\Omega, \qquad x \in D,$$

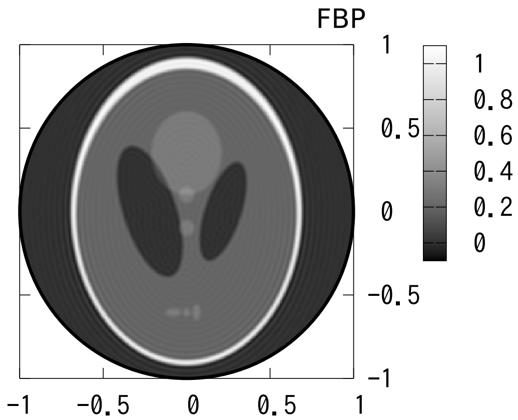
where $\Omega = \arg \omega$, $P_\omega(s) = R\mu(\omega, s)$, and $h(s)$ is the filter function whose Fourier transform approximates $|r|$.

On the other hand, Arbuzov, Bukhgeim and Kazantsev have proposed alternative inversion algorithm based on $A$-analytic theory [1, 2]. They considered the equivalent inverse source problem of the transport equation

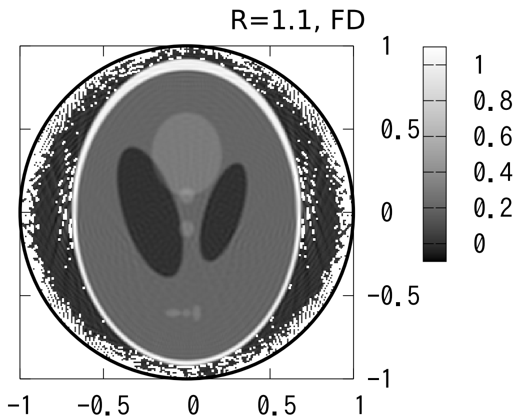
$$\begin{aligned} \xi \cdot \nabla u(x, \xi) &= \mu(x), & (x, \xi) &\in D \times S^1, \\ u(x, \xi) &= 0, & (x, \xi) &\in \Gamma_-, \\ u(x, \xi) &= R\mu(\xi^\perp, x \cdot \xi^\perp), & (x, \xi) &\in \Gamma_+, \end{aligned}$$

where $\Gamma_\pm = \{(x, \xi) \,;\, x \in \partial D, \xi \in S^1, n(x) \cdot \xi \gtrless 0\}$. It is shown that introducing the Fourier expansion $u\big(z, \xi(\theta)\big) = \sum_{n \in \mathbb{Z}} u_n(z) e^{-in\theta}$, the attenuation $\mu$ is reconstructed as

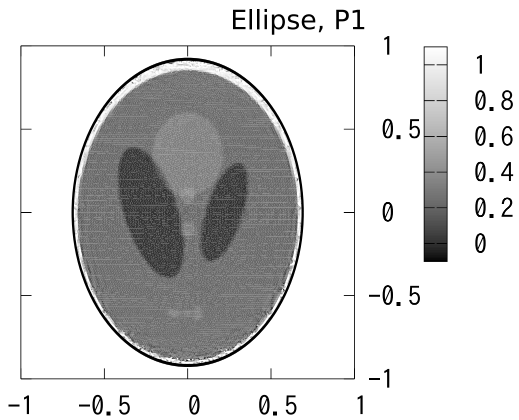
$$\mu(z) = 2 \operatorname{Re} \frac{\partial u_1}{\partial z}(z), \qquad z \in D, \tag{1}$$




(a) Filtered Back Projection with Inverse Radon Transform




(b) Proposed Method, $D$ : Circle with Radius $R = 1.1$, $\partial/\partial z$ : Finite Difference [5]




(c) Proposed Method, $D$ : Ellipse, $\partial/\partial z$ : Numerical Differentiation by Least Square Method [6]

Figure 1: Numerical Reconstructions for Modified Shepp-Logan Phantom [9].

where $x = (x_1, x_2) \in \mathbb{R}^2$ is identified with $z = x_1 + ix_2 \in \mathbb{C}$, and $u_1(z)$, $z \in D$ is obtained by the by the boundary integral of $u_{\text{odd}}(\zeta) = \big(u_1(\zeta), u_3(\zeta), u_5(\zeta), \dots\big)$, $\zeta \in \partial D$ as

$$u_{\text{odd}}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{d\zeta - S^* d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \mathcal{R}\left(\frac{\zeta - z}{\overline{\zeta} - \overline{z}}\right) u_{\text{odd}}(\zeta), \quad z \in D, \tag{2}$$

where $\mathcal{R}(\lambda) = (\lambda I - S^*)^{-1}$ and $S^*(u_1, u_3, u_5, \dots) = (u_3, u_5, u_7, \dots)$. In the formula, the resolvent $\mathcal{R}$ might cause numerical instability. We show a regularization scheme as truncation of high-frequency modes based on the expression given by Finch [3]. A strategy of optimal choice of regularization parameters is also exhibited by calculating the norm of $u_0$ which does not explicitly appear in the reconstruction procedure.

Figures 1 show the numerical realization by (1) and (2). The results by the proposed method, (b) and (c), exhibit similar accuracy as FBP shown in (a). Details are discussed in [5] and [6].

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