Spectral analysis on the elastic Neumann–Poincaré operator *

Daisuke Kawagoe[†]

AMS subject classifications. 35J47 (primary), 35P05 (secondary)

Key words. Neumann–Poincaré operator, Lamé system, polynomial compactness, essential spectrum, corner singularity

1 Introduction

The elastic Neumann-Poincaré (eNP) operator is a boundary integral operator that appears naturally when we solve classical boundary value problems for the Lamé system using layer potentials. Recently, there is rapidly growing interest in the spectral properties of the eNP operator in relation to cloaking by anomalous localized resonance (CALR). Anomalous localized resonance occurs at the accumulation point of eigenvalues, which motivates us to investigate the spectral structure of the eNP operator.

The Lamé system, a system of equations of linear elasticity, is described by

$$\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u = f,$$

where $u = (u_1, \ldots, u_d)$ $(d = 2, 3)$ is the displacement, (λ, μ) are the Lamé constants, and f is the force term. In what follows, we assume that the pair of constants (λ, μ) satisfies the strong convexity condition:

$$\mu > 0, \quad d\lambda + 2\mu > 0.$$

Let $\mathbb{C} = (C_{ijkl})_{i,j,k,l=1}^d$ be the isotropic elasticity tensor corresponding to (λ, μ), namely,

$$C_{ijkl} := \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

Also, let $\widehat{\nabla} u$ be the symmetric gradient of a vector-valued function u, namely,

$$\widehat{\nabla} u := \frac{1}{2} \left(\nabla u + (\nabla u)^T \right),$$

where $(\nabla u)^T$ is the transpose of the matrix ∇u. Then, the Lamé system is also described as

$$\mathcal{L}_{\lambda,\mu} u = \nabla (\mathbb{C} \widehat{\nabla} u) = f.$$

We see what CALR is. Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary. Let $a(x)$ be a function in $\mathbb{R}^d \backslash \partial\Omega$ such that

$$a(x) = \begin{cases} k, & x \in \Omega, \\ 1, & x \in \mathbb{R}^d \backslash \overline{\Omega}, \end{cases}$$

where k is a complex number. We consider the following transmission problem:

$$\begin{cases} \nabla (a(x) \mathbb{C} \widehat{\nabla} u) = f \text{ in } \mathbb{R}^d, \\ |u(x)| = O(|x|^{1-d}) \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

*This work was supported by NRF grants No. 2016R1A2B4011304 and 2017R1A4A1014735.

†Graduate School of Informatics, Kyoto University (d.kawagoe@acs.i.kyoto-u.ac.jp)

where f is a function compactly supported in $\mathbb{R}^d \backslash \overline{\Omega}$ and satisfies

$$\int_{\mathbb{R}^d} f \, dx = 0.$$

Let

$$E(u) := \int_\Omega \widehat{\nabla} u : \mathbb{C} \widehat{\nabla} u \, dx = \int_\Omega \lambda |\nabla \cdot u|^2 + (d-1)\mu |\widehat{\nabla} u|^2 \, dx.$$

Here, $A : B = \sum_{ij} a_{ij} b_{ij}$ for two matrices $A = (a_{ij})$ and $B = (b_{ij})$. We assume that $k = k_0 + i\delta$ in (1.1) for $\delta > 0$, where

$$k_0 := -\frac{\lambda + 3\mu}{\lambda + \mu}.$$

Let u_δ be the solution to the problem (1.1). Then, CALR is characterized by two conditions:

$$\begin{cases} \limsup_{\delta \downarrow 0} \delta E(u_\delta) = \infty, \\ |u_\delta(x)| < C, \quad |x| > R \end{cases}$$

for some positive constants C and R independent of δ.

Let $v_\delta := u_\delta / \sqrt{\delta E(u_\delta)}$. Then, we have $\delta E(v_\delta) = 1$ and $|v_\delta(x)| \to 0$ as $\delta \downarrow 0$ for $|x| > R$. In other words, we cannot observe the displacement outside a ball.

It is worth mentioning that, in the case that Ω is a ball or an ellipse in $\mathbb{R}^2$, CALR occurs when the support of f locates in a suitable area, and that we can replace k_0 by $1/k_0$ to see CALR [1].

Let $\mathbf{\Gamma}(x) = (\Gamma(x))_{i,j=1}^d$ be the fundamental solution to the Lamé system associated with the Lamé constants (λ, μ), namely,

$$\Gamma_{ij}(x) := \begin{cases} \dfrac{\alpha_1}{2\pi} \delta_{ij} \log |x| - \dfrac{\alpha_2}{2\pi} \dfrac{x_i x_j}{|x|^2}, & d = 2, \\ -\dfrac{\alpha_1}{4\pi} \dfrac{\delta_{ij}}{|x|} - \dfrac{\alpha_2}{4\pi} \dfrac{x_i x_j}{|x|^3}, & d = 3, \end{cases} \qquad |x| \neq 0,$$

where

$$\alpha_1 := \frac{1}{2} \left(\frac{1}{\mu} + \frac{1}{\lambda + 2\mu} \right), \quad \alpha_2 := \frac{1}{2} \left(\frac{1}{\mu} - \frac{1}{\lambda + 2\mu} \right).$$

Also, for a vector-valued function u, the conormal derivative $\partial_\nu u$ corresponding to the Lamé system is defined by

$$\partial_\nu u := (\mathbb{C} \widehat{\nabla} u) n = \lambda (\nabla \cdot u) n + 2\mu (\widehat{\nabla} u) n,$$

where n is the outward unit normal to $\partial\Omega$. Then, the eNP operator $\mathbf{K}^*$ is defined by

$$\mathbf{K}^*[f](x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_x} \mathbf{\Gamma}(x - y) f(y) \, d\sigma_y, \qquad \text{a.e. } x \in \partial\Omega.$$

Here, we consider the conormal derivative $\partial_{\nu_x} \mathbf{\Gamma}(x - y)$ of the matrix columnwise and p.v. stands for the Cauchy principal value.

Let $\kappa_0 := (k_0 + 1)/2(k_0 - 1)$. Then, κ_0 and $-\kappa_0$ are accumulation points of eigenvalues of $\mathbf{K}^*$ when $\partial\Omega$ is smooth.

2 Main Results

So far, we obtain two results on the spectral structure of the eNP operator $\mathbf{K}^*$ on the energy space $H^{-1/2}(\partial\Omega)^d$.

The first one is polynomial compactness of the eNP operator. This result was obtained by a joint work with Hyeonbae Kang (Inha University).

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with the $C^{1,\alpha}$-smooth boundary for some $\alpha > 0$. Let $\mathbf{K}^*$ be the eNP operator on $\partial\Omega$ corresponding to the pair of Lamé constants (λ, μ). Let $p_3(t) := t(t+\kappa_0)(t-\kappa_0)$, where κ_0 is given by*

$$\kappa_0 := \frac{k_0+1}{2(k_0-1)} = \frac{\mu}{2(\lambda+2\mu)}.$$

Then, $p_3(\mathbf{K}^*)$ is compact on $H^{-1/2}(\partial\Omega)^3$. Moreover, $\mathbf{K}^*(\mathbf{K}^*+\kappa_0 I)$, $\mathbf{K}^*(\mathbf{K}^*-\kappa_0 I)$ and $(\mathbf{K}^*)^2-\kappa_0^2 I$ are not compact on $H^{-1/2}(\partial\Omega)^3$.

From Theorem 2.1 and the spectral mapping theorem, we obtain the following result on the asymptotic behavior of eigenvalues.

Corollary 2.1. *The spectrum of $\mathbf{K}^*$ on $H^{-1/2}(\partial\Omega)^3$ consists of three non-empty sequences of eigenvalues which converge to 0, κ_0 and $-\kappa_0$, respectively.*

The counterpart of Theorem 2.1 is proved in [1] replacing the polynomial p_3 by $p_2(t) := (t+\kappa_0)(t-\kappa_0)$. Also, Theorem 2.1 was once proved in [2] by assuming C^∞-smoothness on the boundary $\partial\Omega$.

The second one is a characterization of the essential spectrum when Ω is a planar domain with a corner. This result was obtained by a joint work with Eric Bonnetier (Université Grenoble–Alpes), Charles Dapogny (Université Grenoble–Alpes) and Hyeonbae Kang.

Let α be the opening angle of the corner, and assume that $0 < \alpha < 2\pi$. We define a set $\Sigma(k_0, \alpha)$ by

$$\Sigma(k_0,\alpha) := \overline{\{p \in (0, 1-k_0) | \tilde{d}(p,\xi) = 0 \text{ for some } \xi > 0\}},$$

where

$$\begin{aligned}
\tilde{d}(p,\xi) &= 16 d_+(p,\xi) d_-(p,\xi),\\
d_\pm(p,\xi) &= f_{1,\pm}(p,\xi) f_{2,\pm}(p,\xi) + g(p,\xi),\\
f_{1,\pm}(p,\xi) &= \sinh(\alpha\xi)(p-1) \pm \xi \sin\alpha,\\
f_{2,\pm}(p,\xi) &= \sinh((2\pi-\alpha)\xi)(p+k_0) \pm \xi \sin\alpha,\\
g(p,\xi) &= p(p-1+k_0)\sinh^2((\pi-\alpha)\xi).
\end{aligned}$$

Then, we have the following relation.

Theorem 2.2. *Let $0 < \alpha < 2\pi$. Then, we have*

$$\sigma_{ess}(\mathbf{K}^*) = \frac{1}{2} - \frac{1}{1-k_0}\Sigma(k_0,\alpha).$$

A proof of Theorem 2.2 is based on [3], where we discuss the essential spectrum of the NP operator for the Laplace equation.

References

[1] K. Ando, Y.-G. Ji, H. Kang, K. Kim and S. Yu, Spectral properties of the Neumann-Poincaré operator and cloaking by anomalous localized resonance for the elasto-static system, *Eur. J. App. Math.*, 29 (2018), pp. 189-225.

[2] K. Ando, H. Kang and Y. Miyanishi, Elastic Neumann-Poincaré Operators on three dimensional smooth domains: polynomial compactness and spectral structure, Int. Math. Res. Not., 2019 (2017), no. 12, pp. 3883-3900.

[3] E. Bonnetier and H. Zhang, Characterization of the essential spectrum of the Neumann-Poincaré operator in 2D domains with corner via Weyl sequences, *Rev. Mat. Iberoam.*, 35 (2019), no. 3, pp. 925-948.