Inverse Moving Source Problems for (Time-Fractional) Diffusion(-Wave) Equations[†]

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Within the last decade, evolution equations with fractional derivatives have gathered considerable attention among researchers from both theoretical and applied disciplines due to their feasibility in modeling physical processes such as anomalous diffusion in heterogeneous media. Meanwhile, the corresponding inverse problems on determining various ingredients in the equations have also been investigated intensively in view of their mathematical and practical significance. In this talk, we consider the initial-boundary value problem for a (time-fractional) diffusion(-wave) equation

$$\begin{cases} (\partial_t^\alpha + \mathcal{L})u = F & \text{in } \Omega \times (0,T), \\ \begin{cases} u = 0 & \text{if } 0 < \alpha \le 1, \\ u = \partial_t u = 0 & \text{if } 1 < \alpha \le 2 \end{cases} & \text{in } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where ∂_t^α $(0 < \alpha \le 2)$ denotes the Caputo derivative in time, $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, \dots)$ is a bounded domain with a smooth boundary $\partial\Omega$, and $\mathcal{L}$ is a second order elliptic operator in Ω. Here the source term F describes the external force of the system, which models the contaminant source in the context of anomalous diffusion. If F takes the form of separated variables, there have been a lot of existing literature on determining either the temporal or the spatial component (see [3] and the references therein).

Unlike the conventional formulation, this talk is concerned with a class of inverse moving source problem where F depicts some objects moving along orbits. More precisely, we study the following two problems.

Problem 1 (Determination of orbits)**.** *Let u be the solution to* (1) *with $F(\boldsymbol{x}, t) = g(\boldsymbol{x} - \boldsymbol{\gamma}(t))$, and pick N interior points $\boldsymbol{x}^j \in \Omega$ $(j = 1, \dots, N)$. Provided that the source profile $g(\boldsymbol{x})$ is suitably given, determine the source orbit $\boldsymbol{\gamma}(t)$ $(0 \le t \le T)$ by the multiple point observations of u at $\{\boldsymbol{x}^j\}_{j=1}^N \times [0, T]$.*

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Problem 2 (Determination of profiles)**.** *Let u be the solution to* (1) *with*

$$F(\boldsymbol{x},t)=\begin{cases} g(\boldsymbol{x}-\boldsymbol{p}\,t), & 0<\alpha\le 1,\\ g(\boldsymbol{x}-\boldsymbol{p}\,t)+h(\boldsymbol{x}-\boldsymbol{q}\,t), & 1<\alpha\le 2,\end{cases} \tag{2}$$

and $\omega\subset\Omega$ be a suitably chosen nonempty subdomain of Ω. Provided that $\boldsymbol{q},\boldsymbol{q}\in\mathbb{R}^d$ are given constant vectors such that $\mathbf{p}\ne\boldsymbol{q}$, determine the source profile g,h by the partial interior observation of u in $\omega\times(0,T)$.

It reveals that Problem 1 is nonlinear and Problem 2 is linear. In Problem 1, the orbit $\boldsymbol{\gamma}:[0,T]\longrightarrow\mathbb{R}^d$ is assumed to be a smooth vector-valued function with d components, and thus the number N of observation points should basically be no less than d. In Problem 2, we assume the parallel movement of the source(s), and it is required to determine two profiles g and h simultaneously in the case of $1<\alpha\le 2$.

For Problem 1, we basically restrict the unknown $\boldsymbol{\gamma}$ in the admissible set

$$\mathcal{U}:=\{\boldsymbol{\gamma}\in(C^\infty[0,T])^d\mid \boldsymbol{\gamma}(0)=\mathbf{0},\ \|\boldsymbol{\gamma}\|_{C[0,T]}\le\varepsilon,\ \|\boldsymbol{\gamma}'\|_{C[0,T]}\le K\}$$

with positive constants ε,K. Especially, the restriction $\|\boldsymbol{\gamma}\|_{C[0,T]}\le\varepsilon$ means that $\boldsymbol{\gamma}$ is a localized moving source. In this situation, we pick the minimum necessary d observation points $\{\boldsymbol{x}^j\}_{j=1}^d$ and make the following key assumption:

$$\left|\begin{pmatrix}\nabla g(\boldsymbol{y}^1) & \nabla g(\boldsymbol{y}^2) & \cdots & \nabla g(\boldsymbol{y}^d)\end{pmatrix}^{-1}\right|\le C,\quad \forall\,\boldsymbol{y}^j\in\overline{B_\varepsilon(\boldsymbol{x}^j)},\ j=1,\dots,d. \tag{3}$$

In other words, we assume that the matrix $\begin{pmatrix}\nabla g(\boldsymbol{y}^1) & \cdots & \nabla g(\boldsymbol{y}^d)\end{pmatrix}$ is invertible for all $(\boldsymbol{y}^1,\dots,\boldsymbol{y}^d)\in\prod_{j=1}^d\overline{B_\varepsilon(\boldsymbol{x}^j)}$. Then we can state the Lipschitz stability and uniqueness results for Problem 1.

Theorem 1 (see [1])**.** *Fix $\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathcal{U}$ and denote by u_1 and u_2 the solutions to* (1) *with $\boldsymbol{\gamma}=\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}=\boldsymbol{\gamma}_2$, respectively. If the set of observation points $\{\boldsymbol{x}^j\}_{j=1}^d$ satisfies* (3)*, then there exists a constant $C>0$ depending on g, $\{\boldsymbol{x}^j\}_{j=1}^d$ and $\mathcal{U}$ such that*

$$\|\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2\|_{C[0,T]}\le C\sum_{j=1}^d\|\partial_t^\alpha(u_1-u_2)(\boldsymbol{x}^j,\,\cdot\,)\|_{C[0,T]}.$$

Especially, $u_1=u_2$ at $\{\boldsymbol{x}^j\}_{j=1}^d\times[0,T]$ implies $\boldsymbol{\gamma}_1=\boldsymbol{\gamma}_2$ on $[0,T]$.

The key to proving the above theorem is the reduction of the original problem to a vector-valued Volterra integral equation of the second kind with respect to the difference $\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2$. To this end, we employ the frequently used Mittag-Leffler functions and a newly established fractional Duhamel's principle to obtain the representation of solutions to (1). On the other hand, provided that the key assumption (3) holds globally in Ω, we can remove the localization condition $\|\boldsymbol{\gamma}\|_{C[0,T]}\le\varepsilon$ and obtain a uniqueness result at the cost of very dense observation points.

For Problem 2, we state the uniqueness result as follows.

Theorem 2 (see [2])**.** *Let u be the solution to* (1) *with F given by* (2)*. For $\alpha \in (0,2]\setminus\{1\}$, assume that $\omega \subset \Omega$ is a nonempty subdomain such that $\partial\omega \supset \partial\Omega$.*

(a) *Let $0 < \alpha \le 1$ and $g \in H_0^1(\Omega)$. Then $u = 0$ in $\omega \times (0,T)$ implies $g \equiv 0$ in Ω.*

(b) *Let $1 < \alpha \le 2$ and $g, h \in H^2(\Omega) \cap H_0^1(\Omega)$. For $\alpha = 2$, further assume $T > 2\mathrm{diam}(\Omega)$. Then $u = 0$ in $\omega \times (0,T)$ implies $g = h \equiv 0$ in Ω.*

In the above theorem, the assumption $\partial\omega \supset \partial\Omega$ means that $\overline{\omega}$ covers the whole boundary $\partial\Omega$, which is only absent in the case of $\alpha = 1$. For $\alpha = 2$, the observation duration T is additionally assumed to be sufficiently long due to the finite propagation speed of wave. The key to proving Theorem 2 is a reductions to the inverse problem on the determination of the initial condition by introducing auxiliary functions

$$v := \begin{cases} J^{1-\alpha}(\partial_t + \boldsymbol{p} \cdot \nabla)u, & 0 < \alpha \le 1, \\ J^{2-\alpha}(\partial_t + \boldsymbol{p} \cdot \nabla)(\partial_t + \boldsymbol{q} \cdot \nabla)u, & 1 < \alpha \le 2, \end{cases}$$

where J^β denotes the Riemann-Liouville integral operator. Then we can turn to a unique continuation argument to conclude the vanishing of the initial condition for v to complete the proof.

References

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[2] Y. Liu, G. Hu and M. Yamamoto, Inverse moving source problems for fractional diffusion(-wave) equations: Determination of profiles, preprint.

[3] Y. Liu, Z. Li and M. Yamamoto, Inverse problems of determining sources of the fractional partial differential equations, Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations, De Gruyter, Berlin, 2019, 411–430.