Identification of coefficients in time-harmonic Maxwell's equations and its application to biomagnetic inverse problems

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Imaging of electrical properties inside human bodies provides us with valuable pathological information since the electrical conductivity and permittivity of cancerous tissues are much different from those of normal tissues [1]. Among several techniques, a method based on magnetic resonance (MR) has attracted attention because the magnetic field can be measured using a usual MRI scanner not only on the surface but also inside the human bodies, which enhances stability of the inverse problem for reconstruction of the electrical properties. The governing equations for this biomagnetic inverse problem is the time harmonic Maxwell equations

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \gamma \boldsymbol{E} + \boldsymbol{J}, \quad (1) \\ \nabla \times \boldsymbol{E} &= -i\omega\mu_0 \boldsymbol{H}, \quad (2) \end{aligned}$$

with the Silver-Muller radiation condition

$$\lim_{|\boldsymbol{r}| \to \infty} (\boldsymbol{H} \times \boldsymbol{r} - |\boldsymbol{r}|\boldsymbol{E}) = \boldsymbol{0}, \quad (3)$$

where σ and ϵ are the electrical conductivity and permittivity, respectively, which should be reconstructed inside the human body D, γ is admittivity defined by $\gamma = \sigma + i\omega\epsilon$, ω is the Larmor frequency, μ_0 is permeability assumed to be constant, and $\boldsymbol{J}$ represents currents applied to the coils in the MRI scanner. A measurable quantity using the MRI scanners is a limited component of the magnetic field, that is, the positively rotating magnetic field $H^+ = \frac{1}{2}(H_x + iH_y)$, where the z-axis is parallel to the body axis. Then, our inverse problem is formulated as follows: given H^+ in D, determine γ in D.

In our previous paper [2], we propose a complex analysis-based method that explicitly reconstruct γ from the measured H^+. Assuming that $H_z = 0$ when using a so-called birdcage coil in the MRI scanners and that the spatial derivative of γ with respect to the z-axis is small, we can derive a Dbar equation

$$\bar{\partial} E_z = \omega\mu_0 H^+, \quad (4)$$

from the Faraday law (2), where $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Hence, letting Ω be a region of interest (ROI) in a 2D slice of the human body, for any $w \in \Omega$, the z-component of the electric field can be expressed in terms of the Cauchy integrals as

$$E_z(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{E_z(\zeta)}{\zeta - w} d\zeta - \frac{\omega\mu_0}{\pi} \int\int_{\Omega} \frac{H^+(\zeta)}{\zeta - w} d\xi d\eta, \quad \zeta = \xi + i\eta. \quad (5)$$

Therefore, substituting Equation (5) into the equation

$$4\partial H^+ = i\gamma E_z \quad (6)$$

derived from the z-componen of the Ampere law (1) with $\nabla \cdot \boldsymbol{H} = 0$, we have an explicit formula to reconstruct γ in Ω from the measured H^+ in Ω and the boundary value of γ given as *a priori* information.

However, this formula breaks down when $E_z = 0$. Even when $E_z \neq 0$, large errors are generated when $|E_z|$ is small. To avoid this, we propose to use a physical constraint as a new regularization term for reconstruction of γ. To this end, first, using the components of the Faraday law and Ampere law that have not been used so far for reconstruction, we derive an explicit formula for computing $H^- = \frac{1}{2}(H_x - iH_y)$ from the measured H^+, with which H_x and H_y are separately obtained. Then we derive a partial differential equation (PDE) for γ shere H_x and H_y are included as its coefficients. Since this PDE is a constraint for γ, it can be used in a regularization term for reconstruction of γ. Numerical simulations will be shown in which γ is well reconstructed with the regularization term even where $|E_z|$ is small.

References

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[2] Nara, T. and Fushimi, M., An explicit reconstruction method for magnetic resonance electrical property tomography based on the generalized Cauchy formula, Inverse Problems, 33, 105005, 2017.