Cohomological non-rigidity of eight-dimensional complex projective towers

Shintarō KUROKI and DongYoup SUH

Abstract. A complex projective tower or simply a CP-tower is an iterated complex projective fibrations starting from a point. In this paper, we classify certain class of 8-dimensional CP-towers up to diffeomorphism. As a consequence, we show that cohomological rigidity is not satisfied by the collection of 8-dimensional CP-towers, i.e., there is a two distinct 8-dimensional CP-towers which have the same cohomology rings.

1. Introduction

Let \( \mathcal{M} \) be a collection of diffeomorphism classes of smooth manifolds and \( H^* \mathcal{M} \) be the isomorphism classes of cohomology rings of manifolds in \( \mathcal{M} \). Let \( H^*: \mathcal{M} \rightarrow H^* \mathcal{M} \) be the map defined by \( M \in \mathcal{M} \mapsto H^*(M; \mathbb{Z}) \). In general, \( H^* \) is not bijective. However, if we restrict the class of manifolds then this map sometimes becomes a bijection; e.g., if \( \mathcal{M} \) is a collection of orientable 2-dimensional manifolds then it is well-known that the map \( H^* \) is bijective. We say such collection \( \mathcal{M} \) is cohomologically rigid or \( \mathcal{M} \) satisfies cohomological rigidity. The problem asking whether the map \( H^*: \mathcal{M} \rightarrow H^* \mathcal{M} \) is bijective or not is called a cohomological rigidity problem. In this paper, we study the cohomological rigidity problem for complex projective towers (or simply a CP-tower) introduced in [8].

A CP-tower of height \( m \) is a sequence of complex projective fibrations

\[
\begin{align*}
C_m & \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{\text{a point}\}
\end{align*}
\]

where \( C_i \) is the \( i \)th stage of the tower. If we forget the tower structure, then we call \( C_i \) an \( (i\text{-stage}) \) CP-manifolds. In [8], we show that the diffeomorphism types of 6-dimensional CP-manifolds are determined by their cohomology rings, i.e., the collection of 6-dimensional CP-manifolds \( CP^6 \) is cohomologically rigid. This is the generalization of the fact that the collection \( GBM^6 \) of 6-dimensional generalized Bott manifolds is cohomologically rigid in [6]. On the other hand, it is known that the collection \( GBM_2^n \) of 2n-dimensional 2-stage generalized Bott manifolds is also cohomologically rigid. The purpose of this paper is to show that the collection \( CP^8_2 \) of 8-dimensional 2-stage CP-manifolds is not cohomologically rigid.

To state our main theorem, let us recall the theorem proved by Atiyah and Rees in [?, (2.8) Theorem]. Let \( \mathcal{VECT}_2(CP^3) \) be the collection of vector bundle isomorphism classes of complex 2-dimensional vector bundles over \( CP^3 \).

Theorem 1.1 (Atiyah-Rees). There exists an injective map \( \phi: \mathcal{VECT}_2(CP^3) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \) such that \( \phi((\alpha, \xi)) = (\alpha(\xi), c_1(\xi), c_2(\xi)) \), where \( c_1(\xi) \) and \( c_2(\xi) \) are the first and the second Chern classes of \( \xi \), and \( \alpha(\xi) \) is a mod 2 element which is 0 when \( c_1(\xi) \) is odd.

By Theorem 1.1, any element in \( \mathcal{VECT}_2(CP^3) \) can be denoted by \( \eta(\alpha, c_1, c_2) \), where \( (\alpha, c_1, c_2) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \) such that \( \alpha \equiv 0 \pmod{2} \) when \( c_1 \equiv 1 \pmod{2} \). On the other hand, it can be seen easily that \( P(\eta(\alpha, c_1, c_2)) \) is diffeomorphic to \( P(\eta(0, c_2 - (c_1^2 - 1)/4)) \) if \( c_1 \equiv 1 \pmod{2} \), and is diffeomorphic to \( P(\eta(\alpha, 0, c_2 - c_1^2/4)) \) if \( c_1 \equiv 0 \pmod{2} \), see Lemma 3.2.
Let $N(u) := P(\eta_{(0, 1, u)})$, and let $\mathcal{N} := \{N(u) \mid u \in \mathbb{Z}\}$. Similarly, let $M_\alpha(u) := P(\eta_{(\alpha, 0, u)})$, and let $\mathcal{M} := \{M_\alpha(u) \mid \alpha \in \{0, 1\}, u \in \mathbb{Z}\}$. We now state the main result of the paper (see Theorem 4.2 for (1) and see Theorem 5.2 for more precise statement of (2)).

**Theorem 1.2.** For the classes $\mathcal{M}$ and $\mathcal{N}$, we have the following.

1. The class $\mathcal{N}$ is cohomologically rigid. In fact, the following are equivalent:
   - (a) $N(u)$ is diffeomorphic to $N(u')$;
   - (b) $u = u'$;
   - (c) $H^*(N(u); \mathbb{Z}) \cong H^*(N(u'); \mathbb{Z})$ as graded rings.

2. The class $\mathcal{M}$ is not cohomologically rigid. In fact, $H^*(M_0(u); \mathbb{Z}) \cong H^*(M_1(u); \mathbb{Z})$ as graded rings for all $u$, but if $\frac{u^{(u+1)}}{12} \in \mathbb{Z}$ then $M_0(u)$ is not diffeomorphic, actually not homotopic, to $M_1(u)$.

The second part of the theorem is proved in Proposition 5.4 by showing that $\pi_6(M_0(u)) \neq \pi_6(M_1(u))$ when $\frac{u^{(u+1)}}{12} \in \mathbb{Z}$.

The organization of this paper is as follows. In Section 2, as examples of $\mathbb{C}P$-towers, we explain when flag manifolds admit the structure of $\mathbb{C}P$-tower. In Section 3, we recall some basic facts from [8]. In Section 4, we show that $\mathcal{N}$ satisfies the cohomological rigidity. In Section 5, we compute the 6-dimensional homotopy group of the elements in some class of $\mathcal{M}$ and show that $\mathcal{M}$ does not satisfy the cohomological rigidity.

**2. Flag manifolds of type $A$ and $C$**

The $\mathbb{C}P$-towers contain many interesting classes of manifolds. In the previous paper [8], we introduce that generalized Bott manifolds or the Milnor surface admits the structure of $\mathbb{C}P$-towers. We first introduce the other two examples of $\mathbb{C}P$-towers. Let $\mathbb{C}PM^n_m$ be the collection of $2n$-dimensional $m$-stage $\mathbb{C}P$-manifolds up to diffeomorphism.

**Example 2.1.** A partial flag manifold $F(d_1, d_2, \ldots, d_k)$, for $0 = d_0 < d_1 < d_2 \cdots d_{k-1} < d_k = n + 1$, is defined by the set of the following partial flags:

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^{n+1},$$

where $V_i$ is a complex subspace of $\mathbb{C}^{n+1}$ with the complex dimension $d_i$. This is well-known to be diffeomorphic to the homogeneous space $U(n + 1)/(U(n_1) \times \cdots \times U(n_k))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \ldots, k$. Denote the partial flag manifold $F(i, i + 1, \ldots, n + 1)$ by $F_i$. In particular, we call $F_1 = F(1, 2, \ldots, n + 1)$ a flag manifold of type $A$ (or a complete flag manifold), and denote it by $F(\mathbb{C}^{n+1})$. We will show that the flag manifold of type $A$ has the structure of a $\mathbb{C}P$-tower with height $n$. We first define the map $p_i : F_i \rightarrow F_{i+1}$ by

$$p_i : \{0\} \subset V_i \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1} \mapsto \{0\} \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}.$$

Because the pull-back of a point in $F_{i+1}$ by $p_i$ can be regarded as the set of codimension one subspaces $V_i \subset V_{i+1}$, $F_i$ is a $Gr_i(V_{i+1})$-bundle over $F_{i+1}$. Here, $Gr_i(V_{i+1})$ is the complex Grassmannian of $i$-dimensional subspaces in $V_{i+1}$, i.e., $F(i, i + 1)$. Because the normal sub-space of a codimension one subspace $V_i \subset V_{i+1}$ is just the line going through the origin, the complex Grassmannian of $i$-dimensional subspaces in $V_{i+1}$ may be regarded as the $i$-dimensional complex projective space $P(V_{i+1}) = (V_{i+1} \setminus \{0\})/\mathbb{C}^*$. By using this fact, it is easy to check that $F_i$ is the projectivization of the tautological bundle over $F_{i+1}$, i.e., $F_i = P(\eta_{i+1})$, where the tautological bundle $\eta_{i+1}$ is the complex $(i+1)$-dimensional vector bundle defined by the following subset in $F_{i+1} \times \mathbb{C}^{n+1}$:

$$\{(0 \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}, x) \mid x \in V_{i+1}\}.$$

Therefore, $F(\mathbb{C}^{n+1})$ has the structure of a $\mathbb{C}P$-tower as follows:

$$\mathbb{C}P \rightarrow F_2 = P(\eta_3) \rightarrow \cdots \rightarrow F_n \simeq \mathbb{C}P^n \rightarrow \{*\}.$$

Hence, the flag manifold of type $A$ is an element of $\mathbb{C}PM^n_m^{2+n}$. 


Example 2.2. Let $(\mathbb{C}^{2n}, \omega)$ be a complex vector space with a symplectic structure $\omega$ given by the skew-symmetric bilinear form defined by the following matrix:

$$
\Omega = \left( \begin{array}{cc} O & I_n \\ -I_n & O \end{array} \right),
$$

where $O$ is the $(n \times n)$-zero matrix and $I_n$ is the $(n \times n)$-identity matrix. Let $V$ be a complex linear subspace in $\mathbb{C}^{2n}$. Define the $\omega$-perpendicular space of $V$ to be the subspace

$$
V^\omega = \{ w \in \mathbb{C}^{2n} \mid \omega(v, w) = v^T \Omega w = 0 \text{ for all } v \in V \}.
$$

Note that $(V^\omega)^\omega = V$ and $\dim V + \dim V^\omega = 2n$. We call $V$ is isotropic (resp. coisotropic) if $V \subset V^\omega$ (resp. $V^\omega \subset V$). A symplectic partial flag manifold $Sp^n F(d_1, d_2, \ldots, d_k)$, for $0 = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k \leq n$, is defined by the set of (isotropic) partial flags

$$
\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k \subset \mathbb{C}^{2n},
$$

where $V_i$ is a complex isotropic subspace of $(\mathbb{C}^{2n}, \omega)$ with the complex dimension $d_i$. It is easy to check that this is equivalent to considering the following set of partial flags:

$$
\{0\} \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \subset \mathbb{C}^{2n},
$$

This is well-known to be diffeomorphic to the homogeneous space $Sp(n)/(U(n_1) \times \cdots \times U(n_k) \times Sp(n_{k+1}))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \ldots, k$ and $n_{k+1} = \frac{1}{2}(\dim V^\omega_k - \dim V_k) = n - d_k$. If $d_k = \dim V_k = n$, i.e., $V_k = V_0^\omega$ (called Lagrangian), then $Sp^n F(d_1, d_2, \ldots, d_k, n)$ is diffeomorphic to $Sp(n)/(U(n_1) \times \cdots \times U(n_k))$. Denote the symplectic partial flag manifold $Sp^n F(1, 2, \ldots, i)$ by $Sp^n F_i$ for $i \geq 1$. In particular, we call $Sp^n F_n = Sp^n F(1, 2, \ldots, n)$ a flag manifold of type $C$ (or a symplectic flag manifold), and denote it by $SpF(\mathbb{C}^{2n})$. We will show that the flag manifold of type $C$ has the structure of a $CP$-tower with height $n$. We first define the map $q_i : Sp^n F_{i+1} \to Sp^n F_i$ by

$$
q_i : \{0\} \subset V_1 \subset \cdots \subset V_{i+1} \subset V_i \subset \mathbb{C}^{2n} \to \{0\} \subset V_1 \subset \cdots \subset V_i \subset \mathbb{C}^{2n}.
$$

The pull-back of a point in $Sp^n F_i$ by $q_i$ can be regarded as the set of isotropic subspaces $V_{i+1}$ in $\mathbb{C}^{2n}$ which contains the isotropic subspace $V_i$ as a codimension one subspace. Note that, for any vectors $v \in V_i^\omega \setminus V_i$, the subspace $V_i \oplus \text{span}_\mathbb{C}(v)$ is an isotropic subspace which contains $V_i$ as a codimension one subspace. Therefore, there exists a one-to-one correspondence between the pull-back of a point in $Sp^n F_i$ by $q_i$ and all complex lines in the quotient vector space $V_i^\omega/V_i \simeq \mathbb{C}^{2n-2i}$, i.e., $Sp^n F_{i+1}$ is a $CP^{2n-2i-1}$-bundle over $Sp^n F_i$. By using this fact, it is easy to check that $Sp^n F_{i+1}$ is the projectivization of the quotient bundle over $Sp^n F_i$, i.e., $Sp^n F_{i+1} = P(\zeta_i^w/\zeta_i)$, where two tautological bundles $\zeta_i^w$ and $\zeta_i$ are defined by the following subsets in $Sp^n F_i \times \mathbb{C}^{2n}$, respectively:

$$
\begin{align*}
\{ \{0\} \subset V_1 \subset \cdots \subset V_i \subset V_i^\omega \subset \cdots \subset V_1 \subset \mathbb{C}^{2n}, x \} & \subset V_i^w \subset \mathbb{C}^{2n}, x \}, \\
\{ \{0\} \subset V_1 \subset \cdots \subset V_i \subset V_i^\omega \subset \cdots \subset V_1 \subset \mathbb{C}^{2n}, x \} & \subset V_i \subset \mathbb{C}^{2n}. 
\end{align*}
$$

Note that $\zeta_i^w$ is the $\mathbb{C}^{2n-i}$-vector bundle and $\zeta_i$ is the $\mathbb{C}^i$-vector bundle; therefore, the quotient bundle $\zeta_i^w/\zeta_i$ is the $\mathbb{C}^{2n-2i}$-vector bundle. Therefore, $SpF(\mathbb{C}^{2n})$ has the structure of a $CP$-tower as follows:

$$
\begin{align*}
SpF(\mathbb{C}^{2n}) = P(\zeta_{n-1}^w/\zeta_{n-1}) & \overset{CP^1}{\longrightarrow} Sp^n F_{n-1} = P(\zeta_{n-2}^w/\zeta_{n-2}) \overset{CP^2}{\longrightarrow} \cdots \\
& \overset{CP^{2n-3}}{\longrightarrow} Sp^n F_1 \simeq CP^{2n-1} \longrightarrow \{ * \}.
\end{align*}
$$

Hence, the flag manifold of type $C$ is an element of $CPM^{2n^2}$.

Remark. As is well-known, both of the flag manifolds $F(\mathbb{C}^{n+1}) \simeq U(n+1)/T^{n+1}$ and $SpF(\mathbb{C}^{2n}) \simeq Sp(n)/T^n$ with $n \geq 2$ do not admit the structure of a toric manifold (see e.g. [2]). On the other hand, $U(2)/T^2 \cong Sp(1)/T^1 \cong CP^1$ is a toric manifold.
Moreover, by computing the generators of flag manifolds of other types \((B_n \ (n \geq 3), \ D_n \ (n \geq 4), \ G_2, \ F_4, \ E_6, \ E_7, \ E_8)\), they do not admit the structure of \(\mathbb{C}P\)-towers, see [1] (or [7] for classical types). Namely, we have the following proposition:

**Proposition 2.3.** Let \(M\) be a flag manifold denoted by \(G/T\), where \(G\) is a compact simple Lie group and \(T\) is its maximal torus. If \(M\) admits the structure of a \(\mathbb{C}P\)-tower, then \(G\) must be a compact Lie group of type \(A\) or \(C\).

The following problem also naturally arises (also see Remark 5).

**Problem 1.** Let \(H^* : \mathbb{C}PM \to H^*\mathbb{C}PM\) be the map defined by taking the cohomology rings. Classify diffeomorphism types of all manifolds in the class \((H^*)^{-1}(H^*(U(n+1)/T^{n+1}))\) and \((H^*)^{-1}(H^*(Sp(2)/T^n))\).

## 3. Some preliminaries

In this section, we recall some basic facts.

### 3.1. Preliminaries from [8].

We first recall some basic facts from [8, Section 2].

Let \(\xi\) be an \(n\)-dimensional complex vector bundle over a topological space \(X\), and let \(P(\xi)\) denote its projectivization. Then, the following formula holds (see [8]):

\[
H^*(P(\xi); \mathbb{Z}) \cong H^*(X; \mathbb{Z})/\langle x^{n+1} + \sum_{i=1}^{n} (-1)^i c_i(\pi^*\xi)x^{n+1-i} \rangle
\]

where \(\pi^*\xi\) is the pull-back of \(\xi\) along \(\pi : P(\xi) \to X\) and \(c_i(\pi^*\xi)\) is the \(i\)th Chern class of \(\pi^*\xi\). Here \(x\) can be viewed as the first Chern class of the canonical line bundle over \(P(\xi)\), i.e., the complex 1-dimensional sub-bundle \(\gamma_{\xi}\) in \(\pi^*\xi \to P(\xi)\) such that the restriction \(\gamma_{\xi}|_{x^{-1}(a)}\) is the canonical line bundle over \(\pi^{-1}(a) \cong \mathbb{C}P^{n-1}\) for all \(a \in X\). Therefore \(\deg x = 2\). Since it is well-known that the induced homomorphism \(\pi^* : H^*(X; \mathbb{Z}) \to H^*(P(\xi); \mathbb{Z})\) is injective, we often abuse the notation \(c_i(\pi^*\xi)\) by \(c_i(\xi)\). The formula (2) is called the Borel-Hirzebruch formula.

In order to prove the main theorem, we often use the following two lemmas.

**Lemma 3.1.** Let \(\gamma\) be any complex line bundle over \(M\), and let \(P(\xi)\) be the projectivization of a complex vector bundle \(\xi\) over \(M\). Then, \(P(\xi)\) is diffeomorphic to \(P(\xi \otimes \gamma)\).

**Lemma 3.2.** Let \(\gamma\) be a complex line bundle, and let \(\xi\) be a 2-dimensional complex vector bundle over a manifold \(M\). Then the Chern classes of the tensor product \(\xi \otimes \gamma\) are as follows.

\[
c_1(\xi \otimes \gamma) = c_1(\xi) + 2c_1(\gamma);
\]

\[
c_2(\xi \otimes \gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).
\]

### 3.2. Atiyah-Rees’s theorem.**

By Theorem 1.1, all of the complex 2-plane bundles over \(\mathbb{C}P^3\) can be denoted by \(\eta(\alpha, c_1, c_2)\) for some \((\alpha, c_1, c_2) \in \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}\). Using Lemma 3.1, its projectivization \(P(\eta(\alpha, c_1, c_2))\) is diffeomorphic to \(P(\eta(\alpha, c_1, c_2) \otimes \gamma)\) for any complex line bundle \(\gamma\) over \(\mathbb{C}P^3\). Moreover, by Lemma 3.2 and the proof of Theorem 1.1 in [7], we also have

\[
\eta(\alpha, c_1, c_2) \otimes \gamma \cong \eta(\alpha, c_1 + 2c_1(\gamma), c_1(\gamma)^2 + c_1(\gamma)c_1 + c_2).
\]

Therefore, we may assume \(c_1 \in \{0, 1\}\). Consequently, in order to classify all \(P(\eta(\alpha, c_1, c_2))\) up to diffeomorphisms, it is enough to classify the following:

\[
M_0(u) = P(\eta(0,0,0,u));
\]

\[
M_1(u) = P(\eta(1,0,0,u));
\]

\[
N(u) = P(\eta(0,1,u)),
\]

where \(u \in \mathbb{Z}\). We denote the class of \(M_0(u)\), \(M_1(u)\) up to diffeomorphism by \(\mathcal{M}\) and that of \(N(u)\) by \(\mathcal{N}\). Then, both classes \(\mathcal{M}\) and \(\mathcal{N}\) are the subclasses of \(\mathcal{CP}M^2\) consisting of 8-dimensional 2-stage \(\mathbb{C}P\)-manifolds.
Finally, in this section, we prove $\mathcal{M} \cap \mathcal{N} = \emptyset$ by comparing their cohomology rings. Namely, we prove the following lemma:

**Lemma 3.3.** Two cohomology rings $\mathcal{H}^*(\mathcal{M}_u)$ and $\mathcal{H}^*(\mathcal{N}(u'))$ are not isomorphic for any $u, u' \in \mathbb{Z}$.

**Proof.** By using the Borel-Hirzebruch formula (2), we also have the cohomology rings with $\mathbb{Z}_2$-coefficient as follows:

$$
\mathcal{H}^*(\mathcal{M}_u; \mathbb{Z}_2) \cong \mathbb{Z}_2[X, Y]/(X^4, uX^2 + Y^2), \quad \text{and}
$$

$$
\mathcal{H}^*(\mathcal{N}(u'); \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/(x^4, u'x^2 + xy + y^2).
$$

Now, the element $uX + Y$ in $\mathcal{H}^2(\mathcal{M}_u; \mathbb{Z}_2)$ satisfies that

$$(uX + Y)^2 = u^2X^2 + 2uXY + Y^2 \equiv uX^2 + Y^2 (= 0) \mod 2$$

However, the squares of all non-zero elements $x, y, x + y$ in $\mathcal{H}^2(\mathcal{N}(u'); \mathbb{Z}_2)$ are not zero because of its ring structure. Hence, $\mathcal{H}^*(\mathcal{M}_u) \neq \mathcal{H}^*(\mathcal{N}(u'))$ for all $u, u' \in \mathbb{Z}$. \hfill \Box

Hence, we have the following corollary:

**Corollary 3.4.** There are no intersections between two classes $\mathcal{M}$ and $\mathcal{N}$.

### 4. Cohomological rigidity of $\mathcal{N}$

In this section, we shall prove the cohomological rigidity of the class $\mathcal{N}$. To show that, it is enough to prove the following lemma.

**Lemma 4.1.** The following two statements are equivalent.

1. $\mathcal{H}^*(\mathcal{N}(u)) \cong \mathcal{H}^*(\mathcal{N}(u'))$
2. $u = u' \in \mathbb{Z}$

**Proof.** Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : \mathcal{H}^*(\mathcal{N}(u)) \cong \mathcal{H}^*(\mathcal{N}(u'))$ where

$$
\begin{align*}
\mathcal{H}^*(\mathcal{N}(u)) &\cong \mathbb{Z}[X, Y]/(X^4, uX^2 + xy + Y^2); \\
\mathcal{H}^*(\mathcal{N}(u')) &\cong \mathbb{Z}[x, y]/(x^4, u'x^2 + xy + y^2).
\end{align*}
$$

Here, we may put

$$
f(X) = ax + by \quad \text{and} \quad f(Y) = cx + dy,
$$

for some $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \epsilon = \pm 1$. By taking its inverse, we also have that

$$
f^{-1}(x) = dxX - beY \quad \text{and} \quad f^{-1}(y) = -aeX + cxY.
$$

Because $f(Y^2 + XY + uX^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$, we have that

$$
\begin{align*}
c^2 - d^2u' &= -ua^2 + b^2u - ac + bdu'; \\
2cd - d^2 &= -2abu + b^2u - ad - bc + bd.
\end{align*}
$$

Because $f(X^4) = 0$ and $f^{-1}(x^4) = 0$, there are the following two cases:

1. $b = 0$
2. $b \neq 0$ and $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u' - 1) = -4d^3 - 6d^2b - 4db^2(1-u) - b^3(2u - 1) = 0$.

If $b = 0$, then $|a| = |d| = 1$. Therefore, by (4), $2c = d - a$, i.e., $c = 0$ if $d = a$ or $c = -a$ if $d = -a$. Because $c^2 - u' = -u - ac$ by (3), we have that $u = u'$.

Assume $b \neq 0$. By the equation $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u' - 1) = 0$, we have $b$ is even. Therefore, together with $ad - bc = \pm 1$, we also have $a$ is odd. We note that the equation $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u' - 1) = 0$ can be written as

$$
(2a - b)(2a^2 - 2ab + b^2 - 2b^2u') = 0.
$$

Because $a$ is odd and $b$ is even, the 2nd factor is not zero; therefore, we have that

$$
b = 2a.
$$
Since \( ad - bc = \pm 1 \), we conclude \((a, b) = \pm (1, 2)\). The same argument applied to the equation 
\[-4d^3 - 6d^2b - 4db^2(1 + u) + b^4(2u - 1) = 0\]
shows that \(-b = 2d\) and \((d, b) = \pm (-1, 2)\). Therefore, \((a, b, d)\) must be either \((1, 2, -1)\) or \((-1, -2, 1)\). Then, \(c = 0, -1\) in the former case while \(c = 0, 1\) in the latter case because \(ad - bc = \pm 1\). In any case, it follows from (3) that \(u' + u = 4uu'\) and this identity holds only when \(u = u' = 0\) since \(u, u' \in \mathbb{Z}\). This completes the case where \(b \neq 0\). \(\Box\)

Therefore, by Theorem 1.1 and Lemma 4.1, we have the following theorem.

**Theorem 4.2.** The following three statements are equivalent.

1. Two spaces \(N(u)\) and \(N(u')\) are diffeomorphic.
2. Two cohomology rings \(H^*(N(u))\) and \(H^*(N(u'))\) are isomorphic.
3. \(u = u' \in \mathbb{Z}\).

In particular, the class \(N\) is cohomologically rigid.

This establishes Theorem 1.2 (1).

### 5. Cohomological non-rigidity of \(\mathbb{CP}M^5\)

In this section, we prove that \(M\) is not cohomologically rigid. We first show the following fact about the cohomology rings of elements in \(M\).

**Lemma 5.1.** The following two statements are equivalent.

1. \(H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))\) where \(\alpha, \alpha' \in \{0, 1\}\).
2. \(u = u' \in \mathbb{Z}\).

**Proof.** Because (2) \(\Rightarrow\) (1) is trivial, it is enough to show (1) \(\Rightarrow\) (2). Assume there is an isomorphism \(f : H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))\) where

\[
H^*(M_\alpha(u)) \cong \mathbb{Z}[X, Y]/(X^4, uX^2 + Y^2);
\]

\[
H^*(M_{\alpha'}(u')) \cong \mathbb{Z}[x, y]/(x^4, u'x^2 + y^2).
\]

We may use the same representation for \(f\) as in the proof of Lemma 4.1. Note that \(f(uX^2 + Y^2) = 0\) and \(f^{-1}(u'x^2 + y^2) = 0\). By using the representation of \(f\), we have the following equations:

\[
ua^2 - uu'b^2 + c^2 - u'd^2 = 0;
\]

\[
ua^2 - uu'b^2 + c^2 - a^2u = 0.
\]

By (5) and (6), we have

\[
c^2 = b^2uu';
\]

\[
ua^2 = u'd^2.
\]

Because \(X^4 = 0\), we also have that

\[ab(a^2 - b^2u') = 0.\]

We first assume \(ab \neq 0\). Then

\[a^2 = b^2u'
\]

by this equation. Together with (7) and (8), we have that

\[c^2b^2 = b^4uu' = b^2a^2u = b^2d^2u' = a^2d^2.
\]

This implies that

\[(ad - bc)(ad + bc) = \epsilon(ad + bc) = 0.\]

Hence, \(ad = -bc\). However this gives a contradiction because \(ad - bc = 2ad = \epsilon = \pm 1\). Consequently, we have \(ab = 0\). Since \(ad - bc = \epsilon\), if \(a = 0\) then \(|b| = |c| = 1\); therefore, we have \(u = u' = \pm 1\) by (7); if \(b = 0\) then \(|a| = |d| = 1\); therefore, we have \(u = u'\) by (8). This establishes the statement. \(\Box\)
Lemma 5.1 says that cohomology rings of $M$ are not affected by $\alpha \in \mathbb{Z}_2$. On the other hand, the goal of this section is to prove the following theorem, i.e., some topological types of $M$ are affected by $\alpha \in \mathbb{Z}_2$.

**Theorem 5.2.** Assume $u(u + 1)/12 \in \mathbb{Z}$. The following three statements are equivalent.

1. Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are diffeomorphic.
2. $(\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z}$.
3. Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent.

In order to prove Theorem 5.2, we first compute the 6-dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.4. Now $M_\alpha(u)$ can be defined by the following pull-back diagram:

$$
\begin{array}{ccc}
M_\alpha(u) & \longrightarrow & EU(2) \times U(2) \mathbb{C}P^1 \\
& \downarrow & \\
\mathbb{C}P^3 & \longrightarrow & BU(2)
\end{array}
$$

Let $p : S^7 \to \mathbb{C}P^3$ be the canonical $S^1$-fibration and $P(\xi_{\alpha, u})$ be the pull-back of $M_\alpha(u)$ along $p$. Namely, we have the following diagram:

$$
\begin{array}{ccc}
P(\xi_{\alpha, u}) & \longrightarrow & M_\alpha(u) & \longrightarrow & EU(2) \times U(2) \mathbb{C}P^1 \\
& \downarrow & \downarrow & \downarrow & \\
S^7 & \longrightarrow & \mathbb{C}P^3 & \longrightarrow & BU(2)
\end{array}
$$

Then, we have the following lemma.

**Lemma 5.3.** For $* \geq 3$, $\pi_*(P(\xi_{\alpha, u})) \cong \pi_*(M_\alpha(u))$.

**Proof.** Because $P(\xi_{\alpha, u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha, u})$ and $M_\alpha(u)$ satisfy the following commutative diagram:

$$
\begin{array}{ccc}
\pi_{*+1}(S^7) & \longrightarrow & \pi_{*}(\mathbb{C}P^1) & \longrightarrow & \pi_{*}(P(\xi_{\alpha, u})) & \longrightarrow & \pi_{*+1}(\mathbb{C}P^4) \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\pi_{*+1}(\mathbb{C}P^3) & \longrightarrow & \pi_{*}(\mathbb{C}P^1) & \longrightarrow & \pi_{*}(M_\alpha(u)) & \longrightarrow & \pi_{*}(\mathbb{C}P^3) & \longrightarrow & \pi_{*+1}(\mathbb{C}P^4)
\end{array}
$$

From the homotopy exact sequence of the fibration $S^1 \to S^7 \to \mathbb{C}P^3$, we have $\pi_*(S^7) \cong \pi_*(\mathbb{C}P^3)$ for $* \geq 3$. Therefore, by using the 5 lemma, we have the statement. $\square$

Now we may prove the following proposition.

**Proposition 5.4.** Assume $u(u + 1)/12 \in \mathbb{Z}$. The following two isomorphisms hold.

1. $\pi_6(P(\xi_{\alpha, u})) \cong \pi_6(M_\alpha(u)) \cong \mathbb{Z}_2$ if $\alpha \equiv u + 1/12 \pmod{2}$
2. $\pi_6(P(\xi_{\beta, u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_2$ if $\beta \not\equiv u + 1/12 \pmod{2}$

**Proof.** We first prove the 1st statement. If $u(u + 1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u + 1)/12 \pmod{2}$, then it follows from [?] that $\xi_{\alpha, u}$ is induced from the rank 2 complex vector bundle over $\mathbb{C}P^4$. Namely, there is the following commutative diagram:

$$
\begin{array}{ccc}
\xi_{\alpha, u} & \longrightarrow & \eta_{(\alpha, 0, u)} & \longrightarrow & \tilde{\mu}_{(\alpha, u)} & \longrightarrow & EU(2) \times U(2) \mathbb{C}P^1 \\
& \downarrow & \downarrow & \downarrow & \\
S^7 & \longrightarrow & \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^4 & \longrightarrow & BU(2)
\end{array}
$$

On the other hand, we have that $\pi_7(\mathbb{C}P^4) \cong \pi_7(S^7) = \{0\}$, by using the homotopy exact sequence for the fibration $S^1 \to S^7 \to \mathbb{C}P^4$. This implies that $\xi_{\alpha, u}$ is the trivial $S^2$-bundle over $S^7$. Therefore,

$$P(\xi_{\alpha, u}) = S^7 \times \mathbb{C}P^1$$
when \( u(u + 1)/12 \in \mathbb{Z} \) and \( \alpha \equiv u(u + 1)/12 \pmod{2} \). Hence, we also have that
\[
\pi_6(M_2(u)) \cong \pi_6(S^7 \times CP^1) \cong \pi_6(CP^3) \cong \mathbb{Z}_{12}.
\]

Next we prove the 2nd statement. Let \( \mu_{\alpha,u} : CP^3 \to BU(2) \) be a continuous map which induces \( \pi_6(S^7 \times CP^1) \) and \( \beta \) be the element in \( \mathbb{Z}_{12} \) which is not equal to \( \alpha \). Let \( x \in CP^3 \) and \( s = \mu_{\alpha,u}(x) \in BU(2) \) be base points. Take a disk neighborhood around \( x \in CP^3 \) and pinch its boundary to a point, i.e., the boundary of \( D^6 \subset CP^3 \) pinches to a point, then we obtain the surjective map
\[
\rho : CP^3 \to CP^3 \vee S^6,
\]
where \( CP^3 \vee S^6 \) may be regarded as the wedge sum with respect to the base points \( x \in CP^3 \) and \( y \in S^6 \). Due to theorem of Atiyah-Rees \([?,?]\), we have \( \eta_{(\beta,0,u)} \neq \eta_{(\alpha,0,u)} \). This implies that the vector bundle \( \eta_{(\beta,0,u)} \) is induced from the following continuous map:
\[
(11) \quad \mu_{\beta,u} : CP^3 \xrightarrow{\rho} CP^3 \vee S^6 \xrightarrow{\nu_u} BU(2)
\]
where \( \nu_u = \mu_{\alpha,u} \vee \kappa \) for the generator \( \kappa \in \pi_6(BU(2), s) \cong \mathbb{Z}_2 \).
Hence, we have the following commutative diagram.

\[
\begin{array}{c}
\begin{array}{ccc}
P(\xi_{\beta,u}) & \xrightarrow{M_{\beta}(u)} & EU(2) \times_U(2) \ x CP^1 \\
S^7 & \xrightarrow{p} & CP^3 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{\rho} & \xrightarrow{\mu_{\beta,u}} & BU(2) \\
CP^3 \vee S^6 & \xrightarrow{\nu_u} & \ |
\end{array}
\end{array}
\]

From the \( CP^1 \)-fibrations \( CP^1 \to P(\xi_{\beta,u}) \to S^7 \) and \( CP^1 \to EU(2) \times_U(2) \ x CP^1 \cong BT^2 \to BU(2) \) in the above diagram (12), there is the following commutative diagram.

\[
\begin{array}{c}
\begin{array}{ccc}
\pi_7(S^7) & \xrightarrow{=} & \pi_6(CP^1) \\
\pi_6(P(\xi_{\beta,u})) & \xrightarrow{} & \pi_6(S^7) = \{0\} \\
\pi_6(BU(2)) & \xrightarrow{\cong} & \pi_6(CP^1) \\
\pi_6(BT^2) = \{0\} & \xrightarrow{} & \pi_6(BU(2)) \cong \mathbb{Z}_2
\end{array}
\end{array}
\]

This diagram shows that the following exact sequence:
\[
(13) \quad \mathbb{Z} \cong \pi_7(S^7) \to \pi_7(BU(2))(\cong \mathbb{Z}_{12}) \to \pi_6(P(\xi_{\beta,u})) \to \{0\}.
\]

In this diagram, the left homomorphism is induced from \( \tilde{\mu} := \mu_{\beta,u} \circ p : S^7 \to BU(2) \), say \( \tilde{\mu} : \mathbb{Z} \to \mathbb{Z}_{12} \). Because the diagram (12) is commutative, we may regard that \( \tilde{\mu} := \mu_{\beta,u} \circ p : S^7 \to BU(2) \) can be defined by passing through the map \( \nu_u : CP^3 \vee S^6 \to BU(2) \), i.e., \( \tilde{\mu} = \nu_u \circ \rho \circ p \). Because \( \nu_u = \mu_{\alpha,u} \vee \kappa \), we also have
\[
\tilde{\mu} = (\mu_{\alpha,u} \vee \kappa) \circ \rho \circ p = (\mu_{\alpha,u} \circ \rho \circ p) \vee (\kappa \circ \rho \circ p).
\]

By the argument when we proved the 1st statement, we see that \( \mu_{\alpha,u} \circ \rho \circ p \) induces the trivial bundle over \( S^7 \), i.e., \( \mu_{\alpha,u} \circ \rho \circ p \) is homotopic to the trivial map. This also implies that there is the following decomposition up to homotopy:
\[
\tilde{\mu} : S^7 \xrightarrow{p} CP^3 \xrightarrow{\rho} CP^3 \vee S^6 \xrightarrow{\pi} S^6 \xrightarrow{\kappa} BU(2),
\]
where \( \pi \) is the collapsing map of \( CP^3 \) to a point. Therefore, we have the following decomposition for the induced map
\[
\tilde{\mu} : \pi_7(S^7) \xrightarrow{\Psi \pi} \pi_7(S^6) \cong \mathbb{Z}_2 \xrightarrow{\kappa} \pi_7(BU(2)) \cong \mathbb{Z}_{12},
\]
where the 1st map is induced from the surjective map \( \Psi = \pi \circ \rho \circ p \). Because \( \Psi \) is surjective, i.e., not homotopic to the trivial map, we have \( \Psi_\#(1) = [12] \) (the generator of \( \pi_7(S^6) \cong \mathbb{Z}_2 \)). Moreover,

\(^1\)This construction induces the free \( \pi_6(BU(2)) \cong \pi_5(U(2)) \cong \mathbb{Z}_2 \) action on \( KSp(CP^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \) (see \([?,?]\)).
because $\kappa \in \pi_6(\text{BU}(2)) \cong \mathbb{Z}_2$ is the generator, i.e., non-trivial map, we have $\kappa([1]_2) = [6]_{12} \in \mathbb{Z}_{12}$.

This shows that $\tilde{\mu}_\#(1) = [6]_{12}$; therefore, $\tilde{\mu}_\#(\pi_7(S^7)) = ([0]_{12}, [6]_{12}) \subset \mathbb{Z}_{12}$.

Consequently, by the exact sequence (13), we have that
\[ \pi_6(P(\xi_3, u)) \cong \pi_6(\text{BU}(2))/\tilde{\mu}_\#(\pi_7(S^7)) \cong \mathbb{Z}_{12}/([0]_{12}, [6]_{12}) \cong \mathbb{Z}_6. \]

By Lemma 5.3, we have the statement. \qed

REMARK. For example, the condition $u(u + 1)/12 \in \mathbb{Z}$ is satisfied when $u = 0$ and $u = 3$. In these cases, by using Proposition 5.4, we have
\[ \pi_6(M_\alpha(0)) \cong \begin{cases} \mathbb{Z}_{12} & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_6 & \text{for } \alpha \equiv 1 \end{cases} \]
and
\[ \pi_6(M_\alpha(3)) \cong \begin{cases} \mathbb{Z}_6 & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_{12} & \text{for } \alpha \equiv 1 \end{cases} \]

On the other hand, the case when $u = 1$ does not satisfy the condition $u(u + 1)/12 \in \mathbb{Z}$. It follows from the cohomology ring of the flag manifold of type $C$ (see e.g. [1] or [7]) that the flag manifold $Sp(2)/T^2$ is one of this case, i.e., $M_\alpha(1)$ or $M_\beta(1)$. However, by using the homotopy exact sequence for the fibration $T^2 \to Sp(2) \to Sp(2)/T^2$ and the computation in [12], we have that
\[ \pi_6(Sp(2)/T^2) \cong \pi_6(Sp(2)) = 0. \]

Therefore, Proposition 5.4 is not true for the case when $u(u + 1)/12 \notin \mathbb{Z}$.

Let us prove Theorem 5.2

PROOF OF THEOREM 5.2. By using Theorem 1.1, (2) $\implies$ (1) is trivial. The statement (1) $\implies$ (3) is also trivial. We claim (3) $\implies$ (2). Assume $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent. Then, $H^*(M_\alpha(u)) \cong H^*(M_\beta(u'))$. Therefore, it follows from Lemma 5.1 that $u = u'$. Moreover, in this case, $\pi_6(M_\alpha(u)) \cong \pi_6(M_\beta(u))$. If $\alpha \equiv \beta \mod 2$, then this gives a contradiction to Proposition 5.4. Hence, $\alpha \equiv \beta \mod 2$. We have (3) $\implies$ (2). This establishes Theorem 5.2. \qed

In summary, by Lemma 5.1 and Theorem 5.2, we have the following corollary:

COROLLARY 5.5. The set of 8-dimensional $\mathbb{C}P$-manifolds does not satisfy the cohomological rigidity.

This establishes Theorem 1.2 (2).

Note that if we restrict the class of 8-dimensional $\mathbb{C}P$-manifolds to the 8-dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds by [5]. On the other hand, the following question seems to be natural to ask for the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ instead of the cohomological rigidity problem.

PROBLEM 2. Is the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ (up to diffeomorphism) determined by their homotopy types? More precisely, are $M_1, M_2 \in \mathcal{CPM}$ diffeomorphic if they have the same homotopy types?

ACKNOWLEDGEMENTS. The authors would like to thank to the anonymous referee for giving us invaluable comments. In particular, the proofs of Lemma 3.3 and Lemma 4.1 are greatly simplified by his suggestions. The first author would like to give heartfelt thanks to Prof. Nigel Ray whose comments and helps to stay in University of Manchester were innumerably valuable. He also would like to thank to Prof. Yael Karshon in University of Toronto and Prof. Takashi Tsuboi in The University of Tokyo for giving him an excellent circumstances to do research. Finally, The first author was partially supported by Grant-in-Aid for Scientific Research (S) 24224002, Japan Society for Promotion of Science, the JSPS Institutional Program for Young Researcher Overseas Visits ” Promoting international young researchers in mathematics and mathematical sciences led by OCAMI ”, and the JSPS Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation ”Deepening and Evolution of Mathematics and Physics, Building
of International Network Hub based on OCAMI”. The second author was supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2013R1A1A2007780).

References


Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail address: kuroki@ms.u-tokyo.ac.jp

Department of Mathematical Sciences, KAIST, 335 Gwahangno, Yuseong-gu, Daejeon 305-701, Republic of Korea.
E-mail address: dysuh@math.kaist.ac.kr