K-theory of toric hyperKähler manifolds

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Okayama
Outline

1 Motivation

2 Basic Construction
   - Construction of toric hyperKähler manifolds
   - Cohomology ring of toric hyperKähler manifolds

3 Our Results
   - K-ring of toric hyperKähler manifolds
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   - K-ring of toric hyperKähler manifolds
Earlier work

- Toric hyperKähler manifolds [BD00] were defined by Bielawski and Dancer who study their topology and geometry.
- The integral cohomology ring of toric hyperKähler manifolds was studied by Konno [K00] who gives a presentation for the cohomology ring.
- Algebraic geometric analogue of toric hyperKähler varieties was developed by Hausel and Sturmfels [HS02] and studied in relation to the geometry of toric quiver varieties.
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Our aim

To study K-theory of toric hyperKähler manifolds and toric hyperKähler varieties

We would like to give a presentation of the K-ring using the combinatorics of the associated hyperplane arrangement.

Our earlier results on the K-ring of smooth projective toric varieties, quasitoric manifolds and torus manifolds used the combinatorics of fan or polytope.

We wished to explore if methods used extend to the setting of toric hyperKähler manifolds.
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Notations

- $N := \mathbb{Z}^n$; $M \simeq \text{Hom}(N, \mathbb{Z})$

- $N' := \mathbb{Z}^m$; $M' := \text{Hom}(N', \mathbb{Z})$.

- Let $\{e_1, \ldots, e_m\}$ be a basis of $N'$ and $\{e_1^*, \ldots, e_m^*\}$ be the dual basis of $M'$.

- $\hat{\alpha} := (\alpha_1, \ldots, \alpha_m) \in M'_\mathbb{R} := M' \otimes\mathbb{Z} \mathbb{R}$.

- Let $v_1, \ldots, v_m$ be nonzero primitive vectors in $N$. 
Motivation
Basic Construction
Our Results

Construction of toric hyperKähler manifolds
Cohomology ring of toric hyperKähler manifolds

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Smooth hyperplane arrangements

- $H_i := \{ x \in M_\mathbb{R} | \langle x, v_i \rangle + \alpha_i = 0 \}$ is a codimension 1 affine subspace in $M_\mathbb{R}$ with a normal oriented vector $v_i$.

- $\mathcal{H} := \{ H_1, \ldots, H_m \}$ is a hyperplane arrangement in $M_\mathbb{R}$.

- $\mathcal{H}$ is simple if each nonempty intersection of $k$ hyperplanes has codimension $k$ and if there are $n$ hyperplanes with nonempty intersection.

- $\mathcal{H}$ is smooth if $\mathcal{H}$ is simple and every $n$ linearly independent vectors from $\{ v_1, \ldots, v_m \}$ span $N$. 

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- $\mathcal{H}$ is smooth if $\mathcal{H}$ is simple and every $n$ linearly independent vectors from $\{ v_1, \ldots, v_m \}$ span $N$. 
Since $\mathcal{H}$ is smooth we have a surjective homomorphism
\[ \rho : N' \longrightarrow N \] where $\rho(e_i) := v_i$ for $1 \leq i \leq m$.

$N'' := \ker(\rho) \cong \mathbb{Z}^{m-n}$ and $M'' = \text{Hom}(N'', \mathbb{Z})$.

We get exact sequences of lattices:

\[
0 \longrightarrow N'' \overset{\iota}{\longrightarrow} N' \overset{\rho}{\longrightarrow} N \longrightarrow 0
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0 \longrightarrow M \overset{\rho^*}{\longrightarrow} M' \overset{\iota^*}{\longrightarrow} M'' \longrightarrow 0 \quad (1)
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We also get the corresponding exact sequences of vector spaces:

\[0 \rightarrow N''_R \xrightarrow{\iota_R} N'_R \xrightarrow{\rho_R} N_R \rightarrow 0\]

\[0 \rightarrow M_R \xrightarrow{\rho_R^*} M'_R \xrightarrow{\iota_R^*} M''_R \rightarrow 0\]

Induced exact sequence of tori:

\[1 \rightarrow G := (S^1)^{m-n} \hookrightarrow T' := (S^1)^m \rightarrow T := (S^1)^n \rightarrow 1\]
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HyperKähler structure on $\mathbb{H}^m$

- Consider $\mathbb{H}^m$ with 3 complex structures $I$, $J$, $K$ induced by multiplication by $i$, $j$ and $k$ respectively satisfying the quaternionic relations.

- The diagonal torus $T' = (S^1)^m \subseteq Sp(m) \subseteq SO(4m)$ acts on $\mathbb{H}^m \cong \mathbb{R}^{4m}$ preserving the Riemannian metric and the Kahler forms $\omega_I$, $\omega_J$, $\omega_K$ corresponding to the complex structures $I$, $J$ and $K$ respectively.

$$\mu = (\mu_I, \mu_J, \mu_K) : \mathbb{H}^m \longrightarrow (M'_R)^3$$
denotes the hyperKähler moment map for the $T'$-action.
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Definition of toric hyperKähler manifold

- This further induces an action of $G \hookrightarrow T'$ on $\mathbb{H}^m$ and $\mu_G := \iota^*_R \circ \mu : \mathbb{H}^m \rightarrow (M'')^3$ is the moment map for the $G$-action on $\mathbb{H}^m$.

- Since $\alpha \neq 0$, $(\alpha, 0, 0)$ is a regular value of $\mu_G$.

- Since $\mathcal{H}$ is smooth, $G$ acts freely on $\mu_G^{-1}(\alpha, 0, 0)$ and $\mu_G^{-1}(\alpha, 0, 0)/G$ is a smooth manifold of dimension $4n$.

- $X := \mu_G^{-1}(\alpha, 0, 0)/G$ is called toric hyperKähler manifold equipped with an action of the $n$-dimensional torus $T = T'/G$ which preserves the hyperKähler structure i.e the induced Riemannian metric and complex structures $I_\alpha, J_\alpha, K_\alpha$. 

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Complex line bundles on $X$

- Let $\mathbb{C}_s$ be the 1-dimensional complex vector space with $G$-action induced by $G \xhookrightarrow{} T' \xrightarrow{p_s} S^1$.

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Cohomology ring presentation

- Let $X$ be a toric hyperKähler manifold
  - $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the associated smooth hyperplane arrangement
- Ideal $J$ in $\mathbb{Z}[x_1, \ldots, x_m]$ generated by
  - $\prod_{s \in l} x_s$ whenever $\bigcap_{s \in l} H_s = \emptyset$, $l \subseteq [1, m]$
  - $\sum_{s=1}^{m} \langle u, v_s \rangle x_s$, $u \in M$.
- **Theorem** (Konno) There is an isomorphism of $\mathbb{Z}$-algebras $\phi : \mathbb{Z}[x_1, \ldots, x_m]/J \rightarrow H^*(X; \mathbb{Z})$ that sends $x_s$ to $c_1(L_s)$ for $1 \leq s \leq m$. 

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  $\phi : \mathbb{Z}[x_1, \ldots, x_m]/J \longrightarrow H^*(X; \mathbb{Z})$ that sends $x_s$ to $c_1(L_s)$ for $1 \leq s \leq m$. 
Let $X$ be a toric hyperKähler manifold

$\mathcal{H} = \{H_1, \ldots, H_m\}$— associated smooth hyperplane arrangement

Ideal $J'$ in $\mathbb{Z}[x_1, \ldots, x_m]$ generated by

- $\prod_{s \in I} x_s$ whenever $\bigcap_{s \in I} H_s = \emptyset, I \subseteq [1, m]$
- $\prod_{s \mid \langle u, v_s \rangle > 0} (1 - x_s)^{\langle u, v_s \rangle} - \prod_{s \mid \langle u, v_s \rangle < 0} (1 - x_s)^{\langle u, v_s \rangle}, u \in M.$

**Theorem** ([U]) There is an isomorphism of $\mathbb{Z}$-algebras

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V. Uma: K-theory of toric hyperKähler manifolds
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V. Uma

K-theory of toric hyperKähler manifolds
Idea of proofs

- Although $X$ is non-compact in general it is homotopy equivalent to its “core” $\text{Core}(X)$ which is a finite union of compact toric submanifolds. ($\text{Core}(X)$ is a strong deformation retract of $X$)

- We can apply the Atiyah Hirzebruch spectral sequence which degenerates in this setting since the integral odd cohomology vanishes.

$$E_2^{p,q} = H^p(X, K^q(pt)) \Rightarrow K^{p+q}(X).$$
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- We show (by methods similar to that used for toric and torus manifolds) that $K^*(X)$ is generated by the isomorphism classes of the complex line bundles whose first Chern classes generate the cohomology ring. (Since $H^2(X; \mathbb{Z})$ generates $H^*(X; \mathbb{Z})$.)

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Motivation
Basic Construction
Our Results

Cotangent bundle of complex projective space

Example

The cotangent bundle of the complex projective space $T^*(\mathbb{CP}^n)$ is a toric hyperKähler manifold associated to the hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_n, H_{n+1}\}$ in $\mathbb{R}^n$ consisting of

$$H_j = \{(a_1, \ldots, a_n) \mid a_j = -1\}$$

for $1 \leq j \leq n$ and $H_{n+1} = \{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 1\}$. 
Example

- $J'$ is the ideal in $\mathbb{Z}[x_1, \ldots, x_{n+1}]$ generated by
  - the monomial $x_1 \cdot x_2 \cdots x_{n+1}$ since $I = [1, n + 1]$ is the only subset such that $H_1 \cap \cdots \cap H_{n+1} = \emptyset$
  - and the $n$ relations $(1 - x_j) - (1 - x_{n+1})$ for $1 \leq j \leq n$ corresponding to the basis $e_1^*, \ldots, e_n^*$.

- $\mathbb{Z}[x] / (1 - x)^{n+1} \longrightarrow K^*(X)$ where $x \mapsto 1 - [L_{n+1}]$ defines an isomorphism of $\mathbb{Z}$-algebras.
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K-ring of $T^*(\mathbb{CP}^n)$

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