Positively curved manifolds with isometric torus actions

Michael Wiemeler (joint with Lee Kennard and Burkhard Wilking)

20th November 2019
A question and a problem.

Question

What are the topological implications of positive sectional curvature?
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**Question**

*What are the topological implications of positive sectional curvature?*

**Problem**

*Classify manifolds admitting metrics of positive sectional curvature.*
Examples of positively curved manifolds.

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- For dim $M > 24$ all known examples are diffeomorphic to $S^n$, $\mathbb{C}P^n$, or $\mathbb{H}P^n$. 
Examples of positively curved manifolds.

- There are only very few examples of manifolds admitting metrics of positive sectional curvature.
- For dim $M > 24$ all known examples are diffeomorphic to $S^n$, $\mathbb{C}P^n$, or $\mathbb{H}P^n$.
- Other examples are known in dimensions 6, 7, 12, 13 and 24.
- These are certain homogeneous spaces and biquotient spaces.

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Topological implications of positive curvature.

For closed manifolds $M$ the following is known:

### Classical results

- **Theorem of Gauß-Bonnet:** $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to $S^2$ or $\mathbb{RP}^2$. 
Topological implications of positive curvature.

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**Classical results**

- Theorem of Gauß-Bonnet: $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to $S^2$ or $\mathbb{RP}^2$.
- Theorem of Synge: $\sec(M^{2n}) > 0 \Rightarrow |\pi_1(M)| \leq 2$.
- Theorem of Bonnet-Myers: $\text{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$. 
Topological implications of positive curvature.

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Classical results

- Theorem of Gauß-Bonnet: $\sec(M^2) > 0 \Rightarrow M$ is diffeomorphic to $S^2$ or $\mathbb{RP}^2$.
- Theorem of Synge: $\sec(M^{2n}) > 0 \Rightarrow |\pi_1(M)| \leq 2$.
- Theorem of Bonnet-Myers: $\text{Ric}(M^n) > 0 \Rightarrow |\pi_1(M)| < \infty$.
- Gromov’s Betti number Theorem: $\sec(M^n) \geq 0 \Rightarrow \sum_i b_i(M) < C(n)$.
Two conjectures.

- There are no invariants which can distinguish positively and non-negatively curved simply connected manifolds.
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**Hopf’s Conjecture I**

*If $M$ is a closed, even-dimensional positively curved manifold, then the Euler characteristic of $M$ is positive.*
Two conjectures.

► There are no invariants which can distinguish positively and non-negatively curved simply connected manifolds.
► But there are the following conjectures:

Hopf’s Conjecture I

If $M$ is a closed, even-dimensional positively curved manifold, then the Euler characteristic of $M$ is positive.

Hopf’s Conjecture II

$S^2 \times S^2$ does not admit a positively curved metric.
Remarks.

The first conjecture would imply that $S^{2n+1} \times S^{2n'+1}$ does not admit a positively curved metric.
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The first conjecture is true in dimensions two and four.
A programme.

Grove’s Programme

Classify simply connected positively curved manifolds with large isometry group first.
The result of Grove and Searle.

Theorem (Grove and Searle 1992)

Let $M^n$ be positively curved and simply connected. Assume that there is an isometric, effective action of a torus $T^d$ with

$$d \geq \left\lfloor \frac{n + 1}{2} \right\rfloor.$$

Then $M$ is diffeomorphic to $S^n$ or $\mathbb{C}P^\frac{n}{2}$.
Wilking’s Theorem.

Theorem (Wilking 2003)

Let $M^n$ be manifold with $\pi_1(M) = 0$, $\sec(M) > 0$, and $n \geq 10$. Suppose that there is an effective isometric action of a $d$-dimensional torus $T^d$ on $M^n$ with

$$d \geq \frac{1}{4} n + 1.$$ 

Then $M^n$ is homeomorphic to $\mathbb{H}P^n_4$ or to $S^n$, or $M$ is homotopy equivalent to $\mathbb{C}P^n_2$. 
Main tool in the proof.

Connectedness Lemma (Wilking 2003)

Assume $\sec(M^n) > 0$. If $N^{n-k} \subset M^n$ is a totally geodesic submanifold, then the inclusion $N^{n-k} \rightarrow M^n$ is $(n - 2k + 1)$-connected.
Further results until 2010.

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- Fang and Rong (2005) gave a homeomorphism classification of positively curved $n$-manifolds with symmetry rank $\left\lfloor \frac{n-1}{2} \right\rfloor$.
- Dessai (2007) gives vanishing results for coefficients of the elliptic genus of a positively curved two-connected manifold with isometric $S^1$-action.
Kennard’s result.

**Theorem (Kennard 2013)**

Assume \( \sec(M^n) > 0 \). If \( n \equiv 0 \mod 4 \) and \( M \) admits an effective, isometric \( T^d \)-action with

\[
d \geq 2 \log_2 n - 2,
\]

then \( \chi(M) > 0 \).
Main tool in proof.

Periodicity Theorem (Kennard 2012)

*Let* $M^n$ with $\sec(M^n) > 0$, $\pi_1(M) = 0$.
*Assume there is a pair of totally geodesic, transversely intersecting submanifolds of codimensions* $k_1 \geq k_2$. If $k_1 + 3k_2 \leq n$, then $H^*(M; \mathbb{Q})$ is 4-periodic.
Main tool in proof.

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Definition

Here $H^*(M)$ is called $k$-periodic, if there is a $e \in H^k(M)$ such that

$$\bigcup e: H^i(M) \to H^{i+k}(M)$$

is an isomorphism for all $0 \leq i \leq \dim M - k$ or $H^*(M) \cong H^*(S^{\dim M})$. 
If $H^*(M; \mathbb{Q})$ is four-periodic and $b_1(M) = b_3(M) = 0$, then $H^*(M; \mathbb{Q})$ is one of the following:

$H^*(S^n; \mathbb{Q})$, $H^*(\mathbb{C}P^n_2; \mathbb{Q})$, $H^*(\mathbb{H}P^n_4; \mathbb{Q})$, or $H^*(S^2 \times \mathbb{H}P^{n-2}_4; \mathbb{Q})$. 


Remarks

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- If $H^*(M; \mathbb{Q})$ is four-periodic, $b_1(M) = 0$ and $n \equiv 0 \mod 4$, then $b_3(M) = 0$. 
Further results since 2013.

▶ Amann and Kennard (2014/2015/2017) further improved the bound in Kennard’s original result.
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First main result.

Theorem (Kennard, W., Wilking 2019)

Assume $\sec(M^n) > 0$ and that there is an isometric effective action of a torus $T^d$ of dimension $d \geq 5$.

Then every component $F$ of $M^T$ has the rational cohomology of $S^m$, $\mathbb{C}P^m$ or $\mathbb{H}P^m$. 
First main result.

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This is first result in this direction where the dimension of the acting torus does not grow with the dimension of the manifold.
Some corollaries.

**Corollary**

*Hopf’s Conjecture I holds for manifolds with isometric $T^5$-actions.*
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**Corollary**

*Hopf’s Conjecture* I holds for manifolds with isometric $T^5$-actions.

**Corollary**

The isometry group of a potential positively curved metric on $S^{2n+1} \times S^{2n'+1}$ has rank at most four.
New tool in proof.

**T-Splitting Theorem**

*If* $T^{2d+1} \rightarrow SO(V)$ *is a faithful representation,*

[Proof or explanation here]

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**T-Splitting Theorem**

If $T^{2d+1} \to SO(V)$ is a faithful representation, then there exists a $d$-dimensional subgroup $H \subset T^{2d+1}$ such that the induced representation $T^{2d+1}/H \to SO(V^H)$ is faithful and has exactly $d+1$ non-trivial, pairwise inequivalent, irreducible subrepresentations.

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Outline of proof I.

Localization in equivariant cohomology, leads to relations between $H^*(M)$ and $H^*(M^T)$. 

Therefore it suffices to find for each component $F \subset M^T$, an invariant submanifold $F \subset P \subset M$ with controlled cohomology ring.

To find $P$ do the following:

For simplicity assume $d \geq 7$.

Let $P$ be a component of $M_H$, where $H$ is as in the $T$-Splitting Theorem (with $V = T \times M$ for $x \in F$).

Then there are four totally geodesic submanifolds of $P$ intersecting pairwise transversely.

Applying the Periodicity Theorem to the two submanifolds of smallest codimension, implies that $P$ has four-periodic cohomology.
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  - Then there are four totally geodesic submanifolds of $P$ intersecting pairwise transversely.
  - Applying the Periodicity Theorem to the two submanifolds of smallest codimension, implies that $P$ has four-periodic cohomology.
Outline of proof II.

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Therefore by classical results one only has to deal with the case that $H^*(P; \mathbb{Q}) = H^*(S^2 \times \mathbb{HP}^n; \mathbb{Q})$. 

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Outline of proof II.

- Computations in equivariant cohomology then show $b_3(P) = 0$.
- Therefore by classical results one only has to deal with the case that $H^*(P; \mathbb{Q}) = H^*(S^2 \times \mathbb{H}P^n; \mathbb{Q})$.
- In this case computations in equivariant cohomology and the connectedness lemma lead to the desired result.
Second main result.

Theorem (Kennard, W., Wilking 2019)

Assume $\text{sec}(M^n) > 0$, $\pi_1(M) = 0$. If $M$ admits an isometric effective equivariantly formal action of $T^d$, with

$$d \geq 8,$$

then $M$ has the rational cohomology of $S^n$, $\mathbb{C}P^n_2$, or $\mathbb{H}P^n_4$. 

Bott's Conjecture asks whether a positively curved manifold is rationally elliptic.

Together with Hopf's Conjecture it would imply that $H_{\text{odd}}(M; \mathbb{Q}) = 0$ in even-dimensions. This in turn implies equivariant formality.
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- Together with Hopf’s Conjecture it would imply that \( H^{\text{odd}}(M; \mathbb{Q}) = 0 \) in even-dimensions. This in turn implies equivariant formality.
A corollary.

Corollary

The isometry group of a potential positively curved metric on $S^{2n} \times S^{2n'}$ and $S^{2n} \times S^{2n'-1}$, $n' \leq n$ has rank at most seven.
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The isometry group of a potential positively curved metric on $S^{2n} \times S^{2n'}$ and $S^{2n} \times S^{2n' - 1}$, $n' \leq n$ has rank at most seven.

For the proof note that all torus actions on these manifolds are equivariantly formal.
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The isometry group of a potential positively curved metric on $S^{2n} \times S^{2n'}$ and $S^{2n} \times S^{2n' -1}$, $n' \leq n$ has rank at most seven.

- For the proof note that all torus actions on these manifolds are equivariantly formal.
- There exist equivariantly non-formal actions on $S^2 \times S^3$. 
Outline of proof I.

Equivariant formality, implies that

$$H^*(M; \mathbb{Q}) \cong H^*_T(M; \mathbb{Q})/(H^0(BT)),$$
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\[ H^*(M; \mathbb{Q}) \cong H^*_T(M; \mathbb{Q}) / (H^0(BT)), \]

and that

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is injective.

► Hence it suffices to determine the image of \( \iota^* \).
Outline of proof II.

Let $M_1 = \{ x \in M; \dim Tx \leq 1 \} = \bigcup_{T^{d-1} \subseteq T^d} M^{T^{d-1}}$. 

Lemma (Chang, Skjelbred 1974) Assume that the $T$-action on $M$ is equivariantly formal. Then for every closed invariant subspace $M_1 \subset X \subset M$, 

$$\iota^* H^*_{T}(X; \mathbb{Q}) = \iota^* H^*_{T}(M; \mathbb{Q}) \subset H^*_{T}(M^{T^{d-1}}; \mathbb{Q})$$

Hence, it suffices to determine the combinatorial structure of $M_1$. 

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Outline of proof III.

In even dimensions $n$, we now have to consider two cases:

▶ $\exists T^7 \subset T^8$ and $F_0 \subset M^{T^7}$, $\dim F_0 \geq 4$.

▶ The $T^8$-action is GKM. (This case is similar to the discussion in Goertsches-W. 2015)
Outline of proof IV.

In the first case. Then:

- Using work of Smith, Bredon, Hsiang-Su,

\[ M_1 \subset \bigcup_{T^5 \subset T^7} N_{T^5} =: X, \]

Using a Mayer-Vietoris argument, we show that
\[ \iota^* H^*_{T^7}(X; \mathbb{Q}) \subset H^*_{T^7}(M; \mathbb{Q}) \]
is isomorphic to a similar algebra constructed for a linear action on some \( S^n, \mathbb{H}P^n, \) or \( \mathbb{C}P^n \), respectively.

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is isomorphic to a similar algebra constructed for a linear action on some \( S^n, \mathbb{H}P^n, \) or \( \mathbb{C}P^n \), respectively.

- The Chang–Skjelbred Lemma now implies the claim.
Thank you!