

On the integral cohomology ring of the Peterson variety

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The (full) flag variety of \mathbb{C}^n :

$$Fl(\mathbb{C}^n) = \{ (V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \mid \dim V_i = i, 1 \leq i \leq n \}$$

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- (ii) the permutohederal variety (\leftarrow a toric manifold).

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- Singular projective variety (except for the small ranks)

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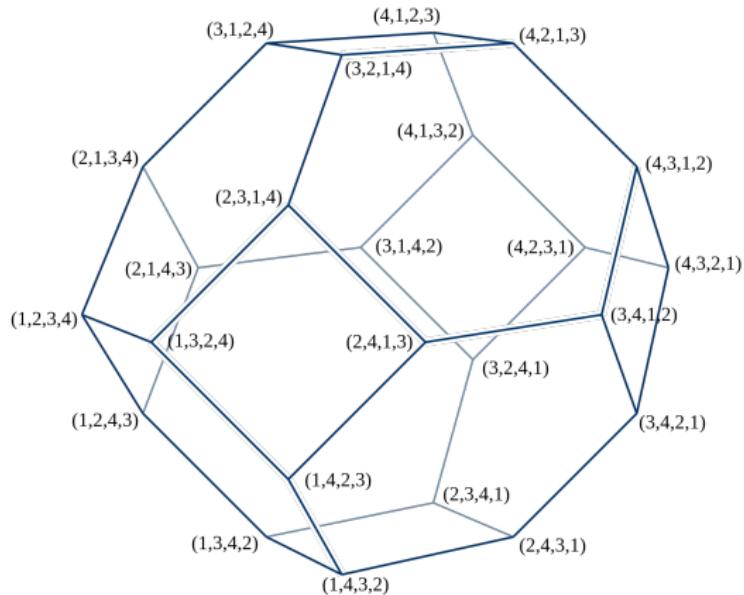
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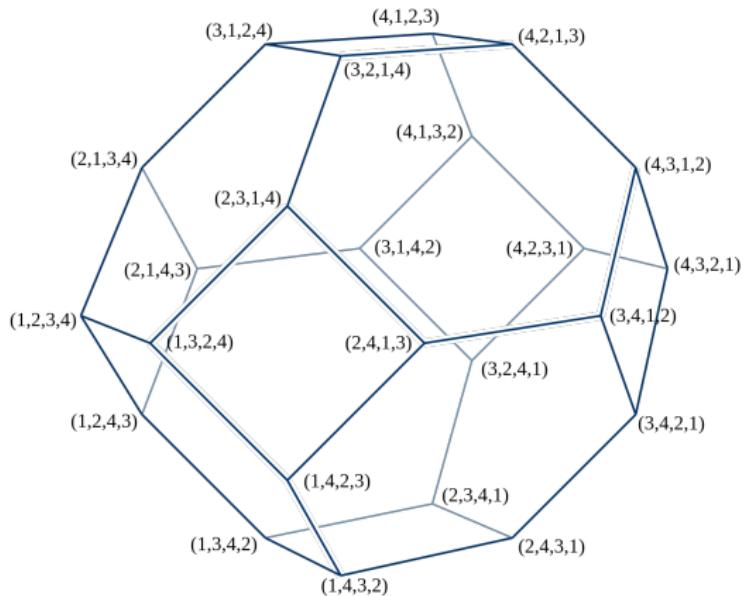
- torus orbit closure in $Fl(\mathbb{C}^n)$ corresponding to permutohedron
- smooth projective (toric) variety

Example : The 3-dimensional permutohedron :



[From Wikipedia (David Eppstein)]

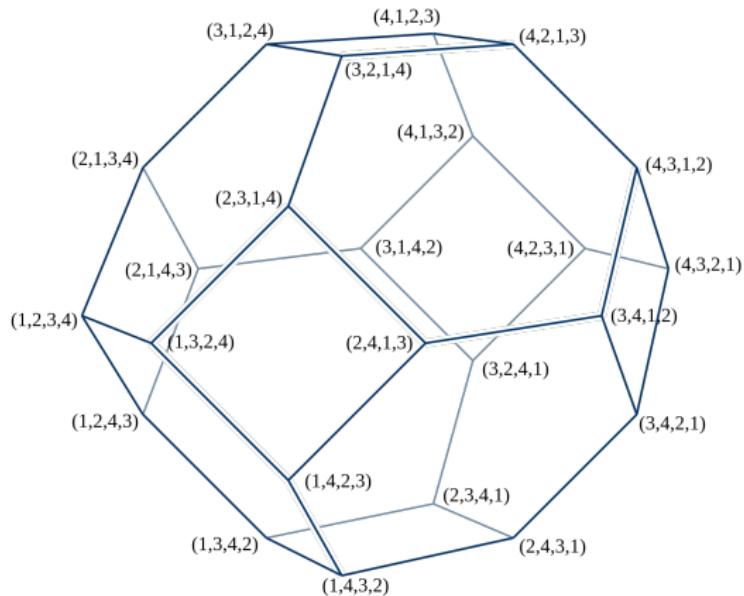
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$$S_4 \curvearrowright \text{Perm} (\subseteq Fl_4)$$

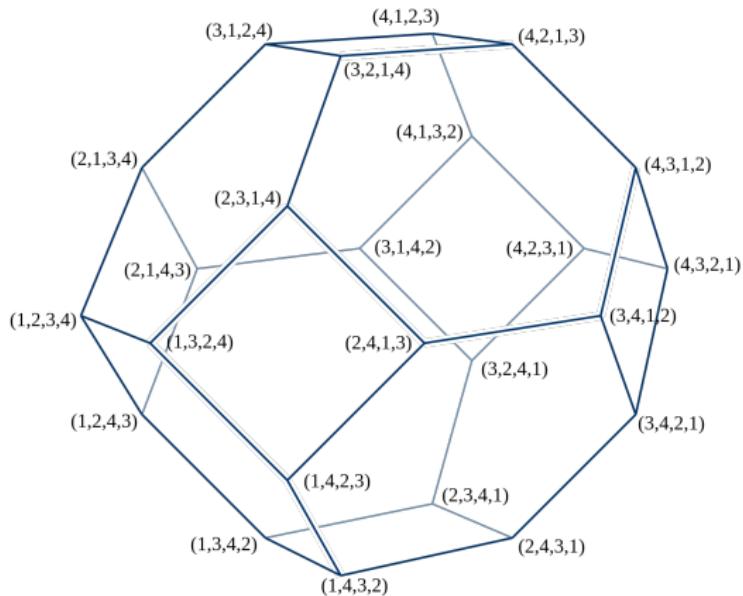
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$S_n \curvearrowright \text{Perm} (\subseteq \text{Fl}_n) \rightsquigarrow S_n \curvearrowright H^*(\text{Perm}; \mathbb{Q})$: representation

The Peterson variety and the permutohedral variety :

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Using $S_n \curvearrowright H^*(Perm; \mathbb{Q})$, we can describe an interesting relation between Pet and Perm.

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where $\alpha_i = -\varpi_{i-1} + 2\varpi_i - \varpi_{i+1}$ and $\varpi_0 = \varpi_n = 0$.

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- Harada-Horiguchi-Masuda

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- Balibanu-Crooks :
→ regular Hessenberg varieties

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To solve this problem, we use a flat degeneration $\text{Perm} \rightsquigarrow \text{Pet}$.

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Thank you for your attention!