Clique complexes of multigraphs

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Plan:

1. Iterative constructions of simplicial complexes: substitutions, vertex inflations.
2. Tournaments in digraphs, cliques in multigraphs, edge inflations of simplicial complexes.
3. Generalizations and open questions.

Iterated wedge construction

Construction of Bahri–Bendersky–Cohen–Gitler

- K : a simplicial complex on vertices $[m] = \{1, \dots, m\}$;
- $(j_1, \dots, j_m)$: an array of positive integers.

Define iterated wedge construction $K(j_1, \dots, j_m)$ with

- Vertex set $J_1 \sqcup \dots \sqcup J_m$ (with $|J_i| = j_i$).
- $A_1 \sqcup \dots \sqcup A_m$ is a simplex iff $\{i \mid A_i = J_i\} \in K$.

Construction has nice properties.

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- If K is a boundary of simplicial polytope then so is $K(j_1, \dots, j_m)$.

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- If K is a sphere then so is $K(j_1, \dots, j_m)$.
- If K is a boundary of simplicial polytope then so is $K(j_1, \dots, j_m)$.
- $\mathcal{Z}_{K(\underline{j_i})}(D^1, S^0) = \mathcal{Z}_K(\underline{(D^{j_i}, S^{j_i - 1})})$, etc.

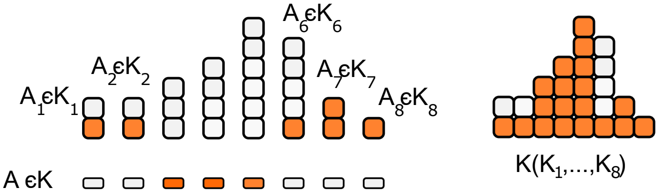
Related construction: substitutions of complexes

Construction of thick substitution

- K : a simplicial complex on vertices $[m] = \{1, \ldots, m\}$;
- $(K_1, \ldots, K_m)$: an array of simplicial complexes on vertex sets $J_1, \ldots, J_m$ resp..

Define new complex $K(K_1, \ldots, K_m)$ with

- Vertex set $J_1 \sqcup \cdots \sqcup J_m$.
- $A_1 \sqcup \cdots \sqcup A_m$ is a simplex iff $\{i \mid A_i \notin K_i\} \in K$.



Thick substitution

Remark: thick substitution generalizes iterated wedges:

$$K(j_1, \ldots, j_m) = K(\partial\Delta^{j_1-1}, \ldots, \partial\Delta^{j_m-1}).$$

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Theorem (A.'13)

- Thick substitution defines an operad structure on the set of all finite simplicial complexes.
- There is a natural homotopy equivalence $|K(K_1,\dots,K_m)|\simeq|K|*|K_1|*\dots*|K_m|$.
- An operation of substitution of polytopes (by Agnarsson'13) corresponds to thick substitution of simplicial complexes.

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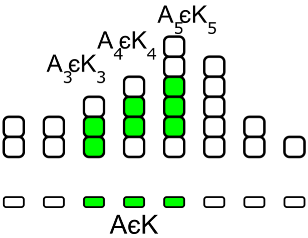
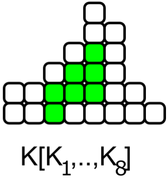
Define new complex $K[K_1, \ldots, K_m]$ with

- Vertex set $J_1 \sqcup \cdots \sqcup J_m$.
- $A_1 \sqcup \cdots \sqcup A_m$ is a simplex iff $A_i \in K_i$ and $\{i \mid A_i \neq \varnothing\} \in K$.





Thin substitution

Remark: thin substitution was used by Abramyan–Panov'19 to construct nontrivial iterated higher Whitehead products in moment angle complexes.

No general good homotopy properties of $K[K_1, \ldots, K_m]$ are known.

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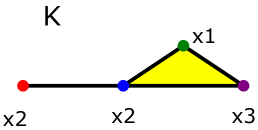
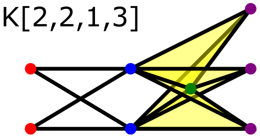
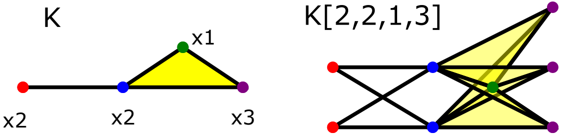
Vertex inflations

Let $n\,\mathrm{pt}$ be a simplicial complex of n disjoint points. A particular case of thin substitution was studied by Björner, Wachs, and Welker. The complex $K[j_1\,\mathrm{pt}, \ldots, j_m\,\mathrm{pt}]$ is called a vertex inflation of K.



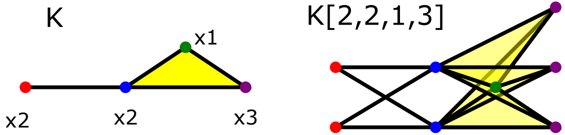


Vertex inflation



Let $\mathcal{J} = (j_1, \ldots, j_m)$ denote the multiplicities.

Vertex inflation



Let $\mathcal{J} = (j_1, \dots, j_m)$ denote the multiplicities.

Theorem (Björner, Wachs, and Welker'05)

1. If Δ^k is a simplex, then $\Delta^k[\mathcal{J}]$ is homotopy equivalent to a wedge of $N(\mathcal{J}) = \prod_i (j_i - 1)$ many spheres S^k.
2. For connected K there is a wedge decomposition
$$|K[\mathcal{J}]| \simeq |K| \vee \bigvee_{I \in K} (\Sigma^{|I|} |\operatorname{link}_K I|)^{\vee N(\mathcal{J}|_I)}.$$
3. If K is Cohen–Macaulay, then $K[\mathcal{J}]$ is also Cohen–Macaulay.

Item 1 follows easily from two observations $\Delta^k[K_1, \dots, K_m] = K_1 * \cdots * K_m$, and $j\,\mathrm{pt} = \bigvee_{j-1} S^0$.

Poset fiber theorem

Wedge decomposition (item 2) follows from Poset Fiber Theorem.

Poset fiber theorem of Björner, Wachs, and Welker

Let $f\colon P \to Q$ be a morphism of posets such that for all $q \in Q$ the fiber $|f^{-1}(Q_{\leqslant q})|$ is $\dim|f^{-1}(Q_{<q})|$-connected. If $|Q|$ is connected, then there is a homotopy equivalence

$$|P| \simeq |Q| \vee \bigvee_{q\in Q} |f^{-1}(Q_{\leqslant q})| * |Q_{>q}|. \tag{1}$$

The proof is based on homotopy colimits.

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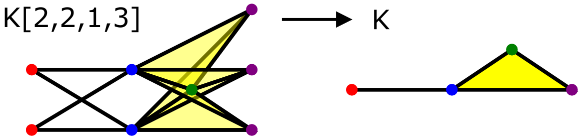
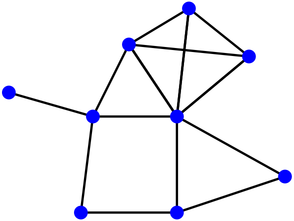
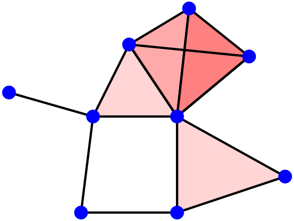
The application idea: notice that there is a “blow down” map which satisfies assumptions of PFT:

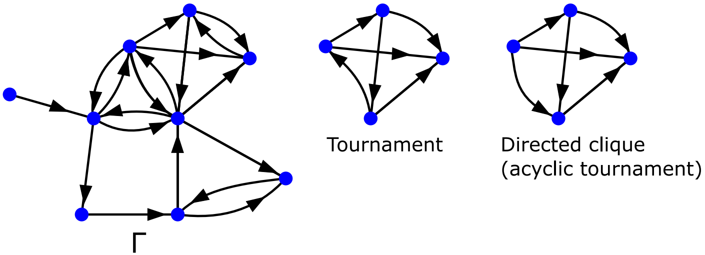


Clique (or flag) complex of a graph





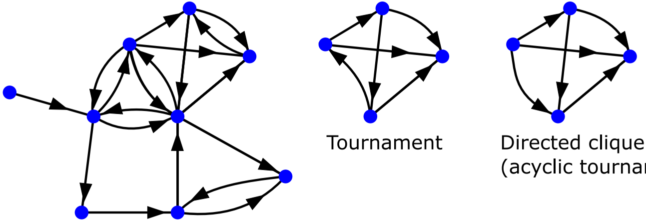
Tournaplex of a digraph



tF.Γ = Tournaplex = poset of all tournaments
dF.Γ = Directed flag complex = poset of all directed cliques

Both $dF.\Gamma$ and $tF.\Gamma$ are simplicial posets.

Tournaplex of a digraph



Γ Tournament   Directed clique (acyclic tournament)

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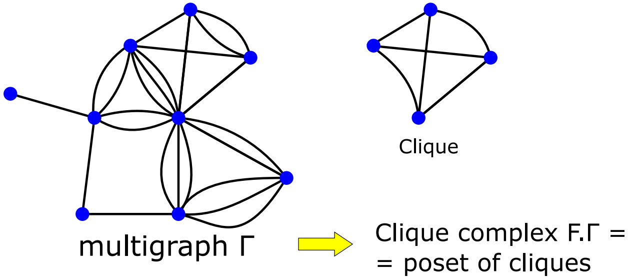
Both $dF.\Gamma$ and $tF.\Gamma$ are simplicial posets.

Introduced and studied in BlueBrain project (Kathryn Hess, Ran Levi, and others).

Used to construct numerical characteristics of brain functional networks.

Clique complex of a multigraph



- Multigraph : multiple edges, no loops.
- A clique : a sub(multi)graph isomorphic to K_n.
- $F.\Gamma$: the simplicial poset of all cliques in Γ.

We want to study topology of $F.\Gamma$.

A generalization: edge inflation

- K : a simplicial complex (or simplicial poset).
- $E(K)$: the edge set of K.
- $\mu\colon E(K) \to \mathbb{Z}_+$: multiplicity function.
- $I \in K$: a simplex, $E(I)$: its edges.

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Construction

For each simplex $I \in K$ we take $\prod_{e\in E(I)} \mu(e)$ many copies of it. Formally, let K_μ be a poset, whose elements have the form (I, c_I), where $I \in K$ and $c_I\colon E(I) \to \mathbb{Z}_+$ maps each edge $e \in E(I)$ to one of the labels $\{1, \ldots, \mu(e)\}$. The order is given by

$$(I, c_I) \leqslant (J, c_J) \text{ iff } I \subseteq J \text{ and } c_J|_{E(I)} = c_I.$$

K_μ is a simplicial poset. We call it the edge inflation of K.

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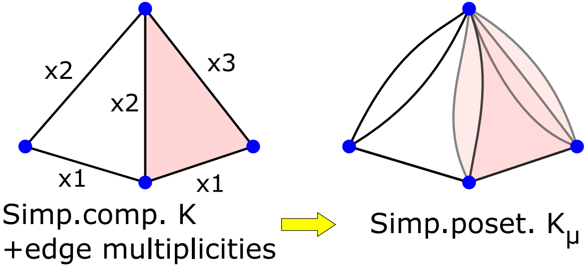

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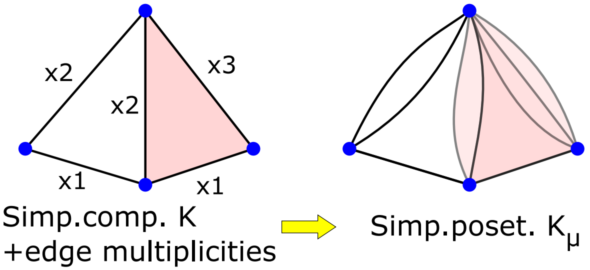

Remark

If Γ is a multigraph, and G is its underlying simple graph (i.e. $\Gamma = G_\mu$), then

$$F.\Gamma = (F.G)_\mu$$

Edge inflations generalize clique complexes of multigraphs.

Homotopy type of edge inflations

Let Δ_V be a simplex on a vertex set V.

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Theorem (A.–Rukhovich'20)

For each multiplicity function the following hold for edge inflated complexes:

1. $(\Delta_V)_\mu$ is homotopy equivalent to a wedge of $n(\mu, V)$ many $S^{|V|-1}$.
2. If K is connected then
$$|K_\mu| \simeq |K| \vee \bigvee_{I \in K, \dim I \geqslant 1} (\Sigma^{|I|} |\operatorname{link}_K I|)^{\vee n(\mu|_I, I)}.$$
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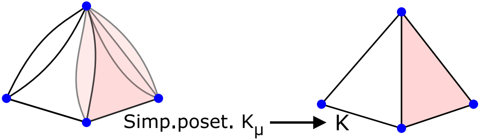
Item 2 follows from Poset Fiber Theorem, since there is the blow-down map:



How many spheres in a wedge?

Theorem, essential part

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The number $n(\mu, V)$ can be computed from Euler characteristic ($n = |V|$):

$$n(\mu, V) = \sum_{J \subseteq V, |J| \geqslant 2} (-1)^{n-|J|} \prod_{e \in \binom{J}{2}} \mu(e) + (-1)^{n-1}(n-1).$$

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Lemma

Let $Y = X_1 \cup_A X_2$. Assume that X_1, X_2 are (homotopy equivalent to) wedges of n-dimensional spheres, and A is a wedge of $(n-1)$-dimensional spheres. Then Y is a wedge of n-dimensional spheres.

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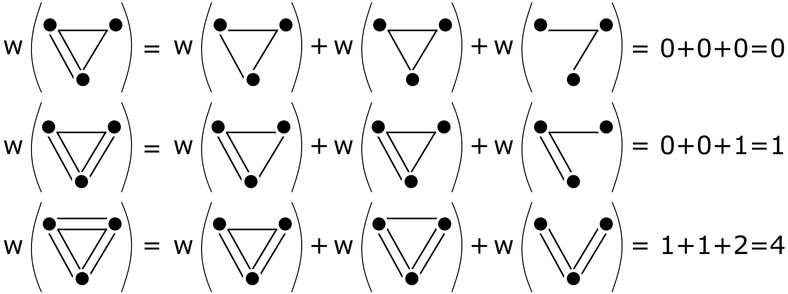
If $w(Z)$ denotes the number of spheres in a wedge Z, then we have $w(Y) = w(X_1) + w(X_2) + w(A)$.

Calculation examples



How many spheres in a wedge?

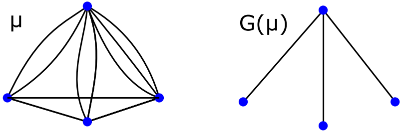
Since $(\Delta_V)_\mu$ is always a wedge of spheres, we can ask

natural questions

- For which μ the complex $(\Delta_V)_\mu$ is contractible ($=$ the wedge of zero many spheres)?
- For which μ the complex $(\Delta_V)_\mu$ is a single sphere?
- Etc.

Characterizations

Let $\mu\colon E(\Delta_V) = \binom{V}{2} \to \mathbb{Z}_+$ be a multiplicity function. Consider the simple graph $G(\mu)$ on V which has an edge $\{i,j\}$ iff $\mu(\{i,j\}) \geqslant 2$.



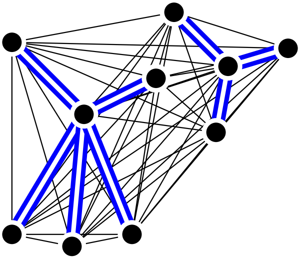
Proposition 0

$(\Delta_V)_\mu$ is contractible iff $G(\mu)$ has an isolated vertex.

Characterizations

We say that μ is a starry sky function if

1. $\mu(e) \in \{1,2\}$ for each $e \in E(\Delta_V)$.
2. All connected components of $G(\mu)$ are star graphs with at least 2 vertices.



Proposition 1

$(\Delta_V)_\mu$ is homotopy equiv. to a single sphere iff μ is a starry sky function.

More infations!

So far we have

- Vertex inflations by Björner, Wachs, Welker.
- Edge inflations, which allow to describe clique complexes of multigraphs.

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Construction

Let K be a simplicial complex (or simplicial poset), and $M\colon K \to \mathbb{Z}_+$ be a multiplicity function. Then we define generalized simplex inflation by

- First, inflate all maximal simplices I, by making $M(I)$ copies of them.
- Next, inflate all simplices below maximal (this produces lots of additional copies of maximal simplices as well).
- Proceed below until you reach vertices.

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- Proceed below until you reach vertices.

We get the same results: (1) Inflation of a simplex produces a wedge of spheres; (2) Homotopy wedge decomposition in general; (3) Cohen–Macaulayness is inherited by inflations.

A very abstract approach and a question

Let K be a simplicial complex. As a poset, it defines a category $\mathrm{cat}(K)$, where

$$I \to J \text{ iff } I \supseteq J.$$

Consider a functor $\mathcal{M}\colon \mathrm{cat}(K) \to \mathrm{FSet}$ to finite sets. We inflate K using $\mathcal{M}$.

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Construction

For each simplex $I \in K$ initialize $|\mathcal{M}(I)|$ its copies. Glue them all together according to the maps $\mathcal{M}(I \supseteq J)$. The resulting space

$$\operatorname{hocolim} \mathcal{M}$$

is called the inflation of K by the functor $\mathcal{M}$.

All previous inflations are particular cases of this construction.

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- Consider K as a finite topological space, open sets are simplicial subcomplexes.
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Conjecture

If $\mathcal{M}$ is a flasque sheaf on K, then

1. For $K = \Delta_V$, $\operatorname{hocolim} \mathcal{M}$ is a wedge of $(|V| - 1)$-dimensional spheres;
2. In general, there is a homotopy wedge decomposition for $\operatorname{hocolim} \mathcal{M}$.

Thank you for listening!

References

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