GKM-Sheaves and Equivariant Cohomology

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Outline

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GKM theory is a collection of techniques for calculating $H_T^*(X)$ and related invariants.

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GKM-sheaves provide a unified framework for these constructions.

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Theorem (The Chang-Skjelbred Theorem)

If $H_T^*(X)$ is a free A-module, then there is a natural exact sequence

$$0 \to H_T^*(X) \xrightarrow{i^*} H_T^*(X_0) \xrightarrow{\delta} H_T^{*+1}(X_1, X_0) \tag{1}$$

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$$0 \to F_0 \to F_1 \to H_T^*(X)$$

where F_0, F_1 are free A-modules.

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The sheaf $\mathcal{F}_X$ is called a *GKM-sheaf* and is defined on a *GKM-hypergraph* Γ_X.

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This determines a functor

$$\Gamma : \text{finite } T\text{-CW complexes} \mapsto \text{GKM-hypergraphs}$$

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GKM-morphisms induce continuous maps in this topology.

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3. $\mathcal{F}(U_e) \cong \mathcal{F}(I(e))$ for all but finitely many $e \in \mathcal{E}$.

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Pushforwards: Given $\phi : \Gamma_1 \to \Gamma_2$, define a functor

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$$Ivt : GKM_G(\Gamma) \mapsto GKM(\Gamma/G), \qquad \mathcal{F} \mapsto (\phi_*(\mathcal{F}))^G$$

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Theorem

There is a canonical morphism of graded A-algebras

$$H_T^*(X) \to H^0(\mathcal{F}_X)$$

which is an isomorphism X if and only if $H_T^*(X)$ is 2-syzygy.

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We show that a section in $\mathcal{F}_M(\mathcal{V})$ extends to $Top(\Gamma)$ if and only if it lies in the kernel of δ.

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Thank you!