GKM-Sheaves and Equivariant Cohomology

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Outline

1. GKM theory
2. The Chang-Skjelbred Theorem
3. Our main result
4. GKM-hypergraphs
5. GKM-sheaves
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GKM-sheaves provide a unified framework for these constructions.
Given a $T$-space $X$, consider the filtration

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_r = X$$

where $X_k$ is the union of orbits of dimension $\leq k$.

Theorem (The Chang-Skjelbred Theorem)

If $H^\ast_T(X)$ is a free $A$-module, then there is a natural exact sequence

$$0 \to H^\ast_T(X) \xrightarrow{i^\ast} H^\ast_T(X_0) \xrightarrow{\delta} H^{\ast+1}_T(X_1, X_0)$$

where $i^\ast$ is induced by inclusion $X_0 \subseteq X$ and $\delta$ is the coboundary map for the pair $(X_1, X_0)$.

By work of Allday-Franz-Puppe, (1) is exact if and only if $H^\ast_T(X)$ is a 2-syzygy, meaning that there exists an exact sequence

$$0 \to F_0 \to F_1 \to H^\ast_T(X)$$

where $F_0, F_1$ are free $A$-modules.
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Furthermore, there is a natural exact sequence

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The sheaf $\mathcal{F}_X$ is called a *GKM-sheaf* and is defined on a *GKM-hypergraph* $\Gamma_X$. 
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This determines a functor

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GKM-morphisms induce continuous maps in this topology.
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3. \( \mathcal{F}(U_e) \cong \mathcal{F}(I(e)) \) for all but finitely many \( e \in \mathcal{E} \).
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**Pushforwards:** Given $\phi : \Gamma_1 \to \Gamma_2$, define a functor

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$$lvt : GKM_G(\Gamma) \mapsto GKM(\Gamma/G), \quad F \mapsto (\phi_*(F))^G$$

by pushing forward and taking invariants.
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$$\mathcal{F}_X(U_V) := H^*_T(V) \cong H^*(V) \otimes_C A$$

and

$$\mathcal{F}_X(U_E) := H^*_T(E) / \text{Tor}_A(H^*_T(E))$$

where we identify vertices $V$ with connected components of $X^T$ and hyperedges $E$ with connected components of $X^{\ker(\alpha_E)}$.

**Theorem**

*There is a canonical morphism of graded $A$-algebras*

$$H^*_T(X) \rightarrow H^0(\mathcal{F}_X)$$

*which is an isomorphism $X$ if and only if $H^*_T(X)$ is 2-syzygy.*
Idea of the proof:
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By Chang-Skjelbred and Allday-Franz-Puppe, we have an exact sequence

$$0 \to H^*_T(X) \xrightarrow{i^*} H^*_T(X_0) \xrightarrow{\delta} H^*_T(X_1, X_0)$$
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By construction $\mathcal{V} \subset \text{Top}(\Gamma_X)$ is an open set and

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We show that a section in $\mathcal{F}_M(\mathcal{V})$ extends to $\text{Top}(\Gamma)$ if and only if it lies in the kernel of $\delta$. 
Thank you!