On the presentations of the commutator subgroup of a right-angled Coxeter group

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Abstract

Right-Angled Coxeter groups

Let Γ be a simplicial graph with $\mathrm{Vert}\Gamma = \{1, 2, \ldots, m\}$.

- We denote by W_Γ right-angled Coxeter group generated by $s_1, \ldots, s_m$ with order 2, where $s_i s_j = s_j s_i$ iff $\{i, j\} \in \mathrm{Edge}\Gamma$.
- Let K_Γ be the flag complex associated with Γ: $K \subset 2^{\mathrm{Vert}\Gamma}$, for $\sigma \subset \mathrm{Vert}\Gamma$, we have $\sigma \in K_\Gamma$ iff σ spans a complete subgraph of Γ.
- For $I \subset \mathrm{Vert}\Gamma$, let $K_{\Gamma,I}$ be the full subcomplex

$$K_{\Gamma,I} = \{\sigma \in K_\Gamma \mid \sigma \subset I\}.$$

A Theorem of Panov and Veryovkin

Let $(a, b) = a^{-1}b^{-1}ab$ be the commutator of the words a and b.

Theorem (Panov-Veryovkin, 2016)

For each

$$I = \{i_k\}_{k=1}^{n} \subset \mathrm{Vert}\Gamma,$$

suppose that we have the splitting

$$K_{\Gamma,I} = \sqcup_{k=1}^{r} K_{\Gamma,I_k}$$

into a disjoint of connected components with $j_k \in I_k$ the smallest index for $k = 1, \ldots, r$. Let $S'_I \subset W_\Gamma$ be the set given by

$$S'_I = \{(s_{i_1}, (s_{i_2}, (s_{i_3}, \ldots, (s_{i_n}, s_{j_k}))\}_{k=1}^{r-1},$$

then $[W_\Gamma, W_\Gamma]$ is generated by $S' = \cup_{I \subset \mathrm{Vert}\Gamma} S'_I$.

Main Theorem

Let $s_I = s_{i_1} s_{i_2} \cdots s_{i_n}$ where $I = \{i_k\}_{k=1}^{n}$ with $i_k < i_{k+1}$ for $k = 1, \ldots, n-1$.

Theorem

For each

$$I = \{i_k\}_{k=1}^{n} \subset \mathrm{Vert}\Gamma,$$

suppose that we have the splitting

$$K_{\Gamma,I} = \sqcup_{k=1}^{r} K_{\Gamma,I_k}$$

into a disjoint of connected components with $j_k \in I_k$ *the smallest index for* $k = 1, \ldots, r$. *Let* $S_I \subset W_\Gamma$ *be the set given by*

$$S_I = \{s_I s_{j_k} (s_{I \setminus \{j_k\}})^{-1}\}_{k=1}^{r-1}$$

then $[W_\Gamma, W_\Gamma]$ *is generated by* $S = \cup_{I \subset \mathrm{Vert}\Gamma} S_I$.

Example

Let Γ be the boundary of a pentagon, namely

$$\mathrm{Edge}\Gamma = \{\{i, i+1\} \mid \ i = 1, \dots, 5 \bmod 5\}.$$

In order that $S_I \neq \emptyset$, $K_{\Gamma,I}$ shall have at least two connected components. That is $I = \{i, i+2\}$ and $I' = \{i, i+1, i+3\}$ for mod 5 integers i, hence $S_I = \{s_i s_{i+2} s_i s_{i+2}\}$ and $S_{I'} = \{s_i s_{i+1} s_{i+3} s_i s_{i+3} s_{i+1}\}$, therefore

$$\begin{aligned} S = \{ & s_1s_3s_1s_3, s_2s_4s_2s_4, s_3s_5s_3s_5, s_4s_1s_4s_1, s_5s_2s_5s_2, \\ & s_1s_2s_4s_1s_4s_2, s_2s_3s_5s_2s_5s_3, s_3s_4s_1s_3s_1s_4, s_4s_5s_2s_4s_2s_5, \\ & s_5s_1s_3s_5s_3s_1\}. \end{aligned}$$

The relation

It can be checked directly that

$$\begin{aligned}1 =&(s_1s_2s_4s_1s_4s_2)(s_2s_5s_2s_5)(s_5s_2s_4s_2s_5s_4)(s_4s_1s_4s_1)\\&(s_5s_1s_3s_5s_3s_1)(s_1s_4s_1s_4)(s_4s_1s_3s_1s_4s_3)(s_3s_5s_3s_5)\\&(s_4s_5s_2s_4s_2s_5)(s_5s_3s_5s_3)(s_3s_5s_2s_5s_3s_2)(s_2s_4s_2s_4)\\&(s_3s_4s_1s_3s_1s_4)(s_4s_2s_4s_2)(s_2s_4s_1s_4s_2s_1)(s_1s_3s_1s_3)\\&(s_2s_3s_5s_2s_5s_3)(s_3s_1s_3s_1)(s_1s_3s_5s_3s_1s_5)(s_5s_2s_5s_2)\end{aligned}$$

in which inside each bracket is a generator or its inverse.

General facts

Let G be a discrete group and Σ be a CW complex with a cellular G-action, namely each element of g maps each cell homeomorphically onto a cell. The following is well known.

Lemma

Suppose that G acts on Σ preserving the orientation of each cell, where Σ is connected and G acts on the 0-skeleton Σ^0 freely and transitively. Let $v_0 \in \Sigma^0$ be a fixed vertex and $E_+ \subset \Sigma^1$ be the set of positively oriented edges. Then G is generated by

$$S = \{1 \neq g \in G \mid v_0 \xrightarrow{e_+} g(v_0)\},$$

where the notation above means that v_0 and $g(v_0)$ are connected by an edge $e_+ \in E_+$ starting from v_0 and ending with $g(v_0)$.

Generators

Let $p\colon \Sigma \to \Sigma/G$ be the quotient map.

Theorem (S from T)

Let $E_+ \subset \Sigma^1$ be the set of positively oriented edges, and suppose the following:

1. *Σ is simply connected with G acting freely;*
2. *$T \subset \Sigma/G$ is a contractible subcomplex containing all vertices of Σ/G, and that T admits a section $\widetilde{T} \subset \Sigma$ so that $p\colon \widetilde{T} \to T$ is a homeomorphim of CW complexes.*

Then G is generated by the set

$$S = \{1 \neq g \in G \mid \widetilde{T} \stackrel{e_+}{\to} g(\widetilde{T})\},$$

where the notation above means that $\widetilde{T}$ and $g(\widetilde{T})$ are connected by an edge $e_+ \in E_+$ starting from a vertex in $\widetilde{T}$ and ending with a vertex in $g(\widetilde{T})$.

The Davis complex

Recall that the Davis complex Σ_Γ associated with the Coxeter group W_Γ is a cube complex with the following properties:

- $\Sigma_\Gamma^0 \cong W_\Gamma$
- Σ_Γ^1 coincides with the Cayley graph of W_Γ
- n-cubes are in one-to-one correspondence with the left cosets W_Γ/W_σ with σ running simplices of K_Γ such that $\mathrm{card}\sigma = n$. Here $W_\sigma = \langle s_i \mid i \in \sigma\rangle \cong (\mathbb{Z}/2)^n$.
- the cube gW_σ is contained in another cube $g'W_{\sigma'}$ if and only if $gW_\sigma \subset g'W_{\sigma'}$ as a set.

Examples

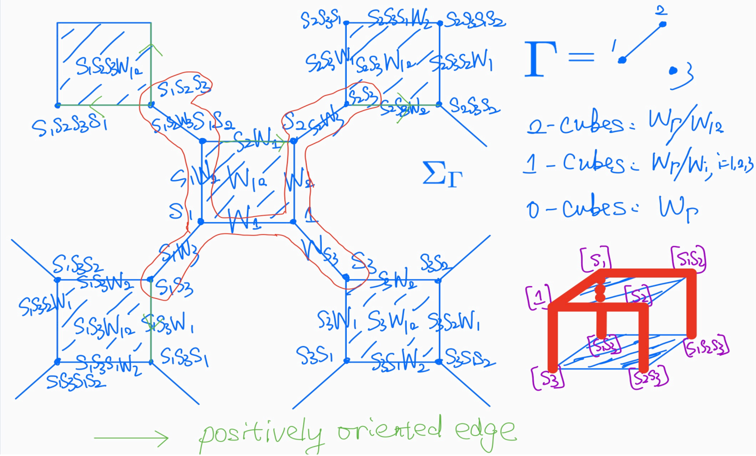

Figure: The Davis Complex Σ_Γ

Facts on Davis complexes

- Topologically $\Sigma_\Gamma = W_\Gamma \times P/\sim$, where $P = Cone|K'_\Gamma|$ with faces $F_i = |\mathrm{Star}(\{i\}, K'_\Gamma)|$, $i = 1, \ldots, m$.
- Σ_Γ is contractible (Gromov, Davis, Moussong,...)
- $[W_\Gamma, W_\Gamma]$ acts freely on Σ_Γ and

$$\Sigma_\Gamma/[W_\Gamma, W_\Gamma] \cong \mathbb{R}\mathcal{Z}_{K_\Gamma}.$$

Contractible subcomplex containing all vertices

Now we define a subcomplex $T \subset \Sigma_\Gamma/[W_\Gamma, W_\Gamma]$.

- $T^0 = \{[s_I] \mid I \subset \mathrm{Vert}\Gamma\} = W_\Gamma/[W_\Gamma, W_\Gamma]$.
- An edge $[s_I]W_i$ is collected in T if and only if $i = \max I$.

As a CW subcomplex of dimension 1, T is contractible. Moreover, a lifting $\widetilde{T} \subset \Sigma_\Gamma$ of T is given by $\widetilde{T}^0 = \{s_I \mid I \subset \mathrm{Vert}\Gamma\}$ and $\mathrm{Edge}\widetilde{T} = \{s_I W_i \mid I \subset \mathrm{Vert}\Gamma,\ i = \max I\}$.

Orientation of Edges

- A general edge connecting $[s_I]$ and $[s_J]$ is positively oriented if it starts from $[s_I]$ and ends with $[s_J]$, where $J \subsetneq I$.
- Since $[W_\Gamma, W_\Gamma]$ acts on Σ_Γ preserving the orientations, we lift the orientations of edges from $\Sigma_\Gamma/[W_\Gamma, W_\Gamma]$ to Σ_Γ.
- Now we apply Theorem S from T as follows: $s_I \in \widetilde{T}$ is connected to $s_I s_j \in g(\widetilde{T})$ by a positively oriented edge iff $j \in I$. We have
$$s_I s_j = g \cdot s_{I \setminus \{j\}},$$
namely $g = s_I s_j (s_{I \setminus \{j\}})^{-1}$.
- Finally we refine these generators by taking only one j from each connected component of $K_{\Gamma,I}$, and drop those trivial elements.

Where do relations come from?

- Recall that $\mathbb{R}\mathcal{Z}_{K_\Gamma} = (\mathbb{Z}/2) \times P/\sim$, where $P = Cone|K'_\Gamma|$ with faces $F_i = |\mathrm{Star}(\{i\}, K'_\Gamma)|$, $i = 1, \ldots, m$.
- Let $X = \mathbb{R}\mathcal{Z}_{K_\Gamma}$ be filtrated that

$$X_n = \bigcup_{\mathrm{Cart} I \leq n} [s_I]P.$$

A new "handle" $[s_I]P \subset X_n$ is attached to X_{n-1} along the union of faces $\cup_{i \in I} F_i \subset \partial P$.

Observation

- When adding a "handle" $[s_I]P$ to X_{n-1} along $\cup_{i\in I} F_i \subset \partial P$, if the union $\cup_{i\in I} F_i$ is contractible, then up to homotopy,

$$X'_{n-1} = [s_I]P \bigcup_{\cup_{i\in I} F_i} X_{n-1} \simeq X_{n-1}.$$

- If $\cup_{i\in I} F_i$ is a disjoint union of contractible spaces, then up to homotopy, X'_{n-1} is the union of X_{n-1} with 1-cells.
- If $\cup_{i\in I} F_i$ contains a loop, then a relation appear after adding $[s_I]P$.

Example

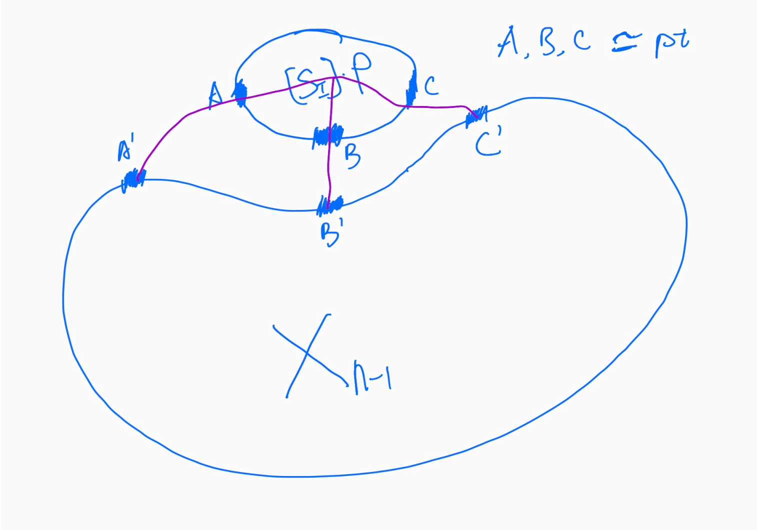

Figure: Homotopy types

Example

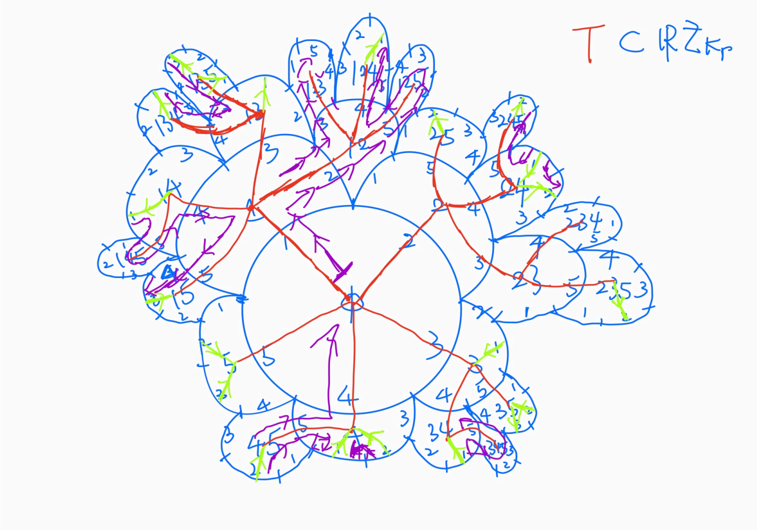

Figure: Looking for relations

Something left

- Davis complex Σ_Γ can be defined in the language of $\square$-set (cubical set without degeneracy). Maybe there is a way to find the relations using cubical sets parallel to the description of homotopy groups using simplicial sets.
- Relations to moment-angle complexes.
- Relations to Bass-Serre Theory.

Thank You!

Thank you very much for your attention.