



On string polytopes I

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Mirror Symmetry for Fano mfds

X : smooth Fano \rightsquigarrow $W: X^\vee \rightarrow \mathbb{C}$ s.t. $QH^*(X) \cong \text{Jac } W$
mirror

E.g. $X = \mathbb{P}^1$

$$QH^*(X) = \mathbb{C}[x, q]/\langle x^2 - q \rangle$$

$$W: \mathbb{C}^* \rightarrow \mathbb{C}$$
$$z \mapsto z + \frac{q}{z}.$$

$$dw = 0 \Leftrightarrow z^2 = q.$$

SYZ mirror symmetry

X^\vee can be constructed by

- gluing $(\mathbb{C}^*)^n$'s
mirror charts
- (partially) compactifying it

More specifically, each Lagrangian torus $LC(M, \omega)$ assigns Laurent poly.

(Special)

$n = \dim L$

$$w_L : (\mathbb{C}^*)^n \rightarrow \mathbb{C} \quad (\text{Foo's disc potential})$$

$\sim T_C^L$

Then (uncorrected) mirror is obtained by gluing those

$M : \text{Fano}$

$M \setminus D$

$D \in |-K_X|$

tori : $\bigcup_L T_C^L$ s.t each gluing map satisfies
cluster var. \rightarrow

$$\text{mutation} \rightarrow g : T_C^L \rightarrow T_C^{L'}, \quad g^* w_{L'} = w_L.$$

sm. Fano var - - - flag var - - - Toric Fano
(spherical)

$\log CT$
(GHKK)

double Bruhat cell
(cluster var.)

$(\mathbb{C}^*)^n$

Symplectic geometry of toric manifolds

Defn (M, ω) : symp. mfd. of $\dim M = 2n$.

(1) $L \subset M$ is **Lagrangian** if . $\dim L = n$
. $\omega|_L = 0$

(2) · D : unit disc with cplx str \bar{J} .

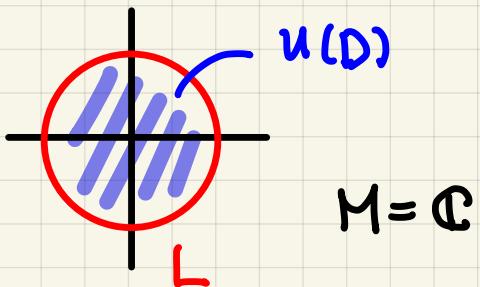
· J : alm. cplx. str. on M

CR eqn.

$u : (D, \bar{J}D) \rightarrow (M, L)$ is **(\bar{J}, J) -hol.** if $J \circ du = du \circ \bar{J}$

or simply hol.

e.g.



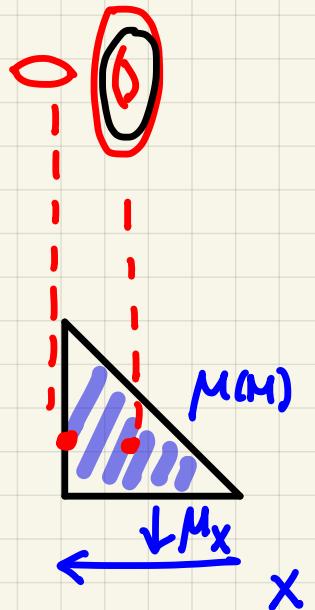
dual Lie alg.
Lie

Example $(\overset{2n}{M}, \omega)$: cpt. Symp. toric mfd. with $\mu: M \rightarrow \Delta \subseteq \mathbb{R}^n = \mathfrak{t}^*$

moment polytope

(1) For $r \in \text{int}(\Delta)$, $L_r := \mu^{-1}(r)$ is a **Lagrangian torus**

(2) Fix $x \in \mathfrak{t}_Z$. Then $\mu_x := \langle \mu, x \rangle$ is Hamiltonian for the S^1 -action gen. by x .

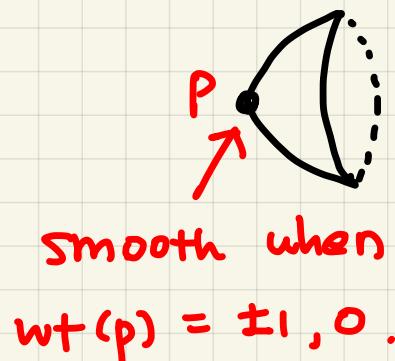


Γ at least # Facets many hol. discs
bounded by L_r .

Take an S^1 -orbit $C \subset L_r$
Flow C along $J\underline{x}$.

Fact: gradient vector field
of μ_x

: (j, J) -hol. discs bounded
by L_r .



Smooth when
 $\text{wt}(p) = \pm 1, 0$.

* If L_1 and L_2 are related by a Symplectomorphism, then

hol. discs bounded by L_1

open GW invariant = # hol. discs bounded by L_2

How to count?

- Each hol. disc represents an element of $\pi_2(M, L)$.
- For $\beta \in \pi_2(M, L)$, define

$$M(L, \beta) := \{ u : (D, \partial D) \rightarrow (M, L) \mid \begin{array}{l} u : \text{hol.} \\ [u] = \beta \end{array} \}$$

moduli space
of hol. discs bounded by L .

$$M_1(L, \beta) := \{ (u, z) : u \in M(L, \beta), z \in \partial D \mid \begin{array}{l} u : \text{hol.} \\ [u] = \beta \end{array} \}$$

moduli space of hol. discs
with one marked point

* $\text{ev} : M_1(L, \beta) \rightarrow L$ evaluation map
 $(u, z) \mapsto u(z)$

Thm. $\dim M_1(L, \beta) = \dim(L) + \mu(\beta) - 2$ where $\mu(\beta)$: Maslov index of β .

∴ Counting is meaningful only when $\mu(\beta) = 2$.

(degree of ev = # hol. discs bdd by L passing through a generic pt in L)
=: n_β (called 'open Gromov-Witten invariant')

* $\mu(\beta)$: Maslov index

(= relative Chern number of a disc)

Potential function For L. monotone Lagrangian torus

$$W_L : \text{Hom}(H_1(L; \mathbb{Z}), \mathbb{C}^*) \rightarrow \mathbb{Q}$$

$\cong (\mathbb{C}^*)^n$

Complex dual torus,

or deformation space of L,

or MC space

$$\rho \mapsto \sum_{\substack{\mu(\beta)=2 \\ \beta \in \pi_2(X, L)}} n_\beta \cdot \rho(d\beta) = \sum n_\beta \cdot z^\beta$$

$$(\mathbb{Z}^n \cong H_1(L; \mathbb{Z}) / \text{Tor})$$

is called a potential function of L.

(wtd counts of Maslov index 2
hol. discs)

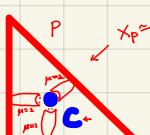
$$\leftarrow x_p \rightarrow \Delta = P: \text{reflexive.}$$

Toric Fano case The central fiber $L_c := \bar{H}(c)$ is the unique monotone Lag. fiber

Thm (Cho - Oh) All disc with $\mu=2$ are regular and

$$W_{L_c} = \sum_{F: \text{facet}} z^{v_F}, \quad v_F : \text{inward primitive normal vector to } F$$

Example



$$W = x + y + \frac{1}{xy}, \quad z^{1,0} := x, \quad z^{0,1} := y$$

Computing potential fnns for "non-toric case"

Def Given a proj. var. X , a flat morphism $\xrightarrow{\text{Sm.}}$

- $\tilde{\pi}^1(\mathbb{C} \setminus \{0\}) \simeq X \times \mathbb{C}^*$ (normal)
- $\tilde{\pi}^1(\omega) \simeq X_0$: toric var.

is called a $\xrightarrow{\text{proj}}$ toric degeneration of X to X_0 .

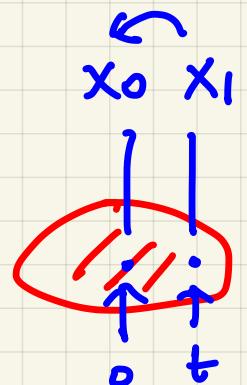
$$\begin{array}{ccc} \tilde{\omega} & & \tilde{\omega}_{FS} \\ \approx & & \\ \mathcal{X} \hookrightarrow \mathbb{P}^N \times \mathbb{C} & \text{s.t.} & \\ \pi \downarrow & \swarrow \text{pr} & \pi(t) \subset \mathbb{P}^N \times \text{fib.} \\ \mathbb{C} & & t \\ + & & \\ \omega_t := \tilde{\omega}|_{\pi^{-1}(t)} & & \end{array}$$

Thm (Maruta - Kaveh) \exists continuous map $\phi : X \rightarrow X_0$ s.t.

ϕ is symplectomorphism on $\overset{\text{surjective}}{\Phi}(\overset{\circ}{\Delta}_0)$
 $\overset{\sim}{=}$
open dense.

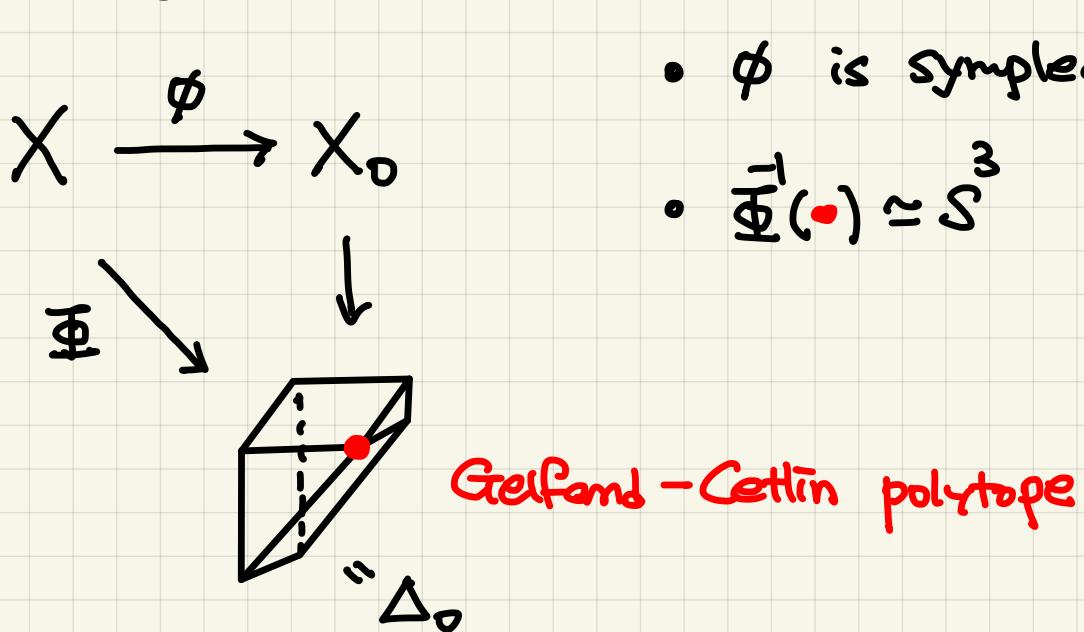
$$\omega_t \quad \omega_0$$

$$\Phi \xrightarrow{\sim} \Phi_0$$



Cor Every generic fiber of Φ is a Lagrangian torus

Example . $X = \mathrm{SL}_3 \mathbb{C} / B$ \exists toric degeneration s.t.



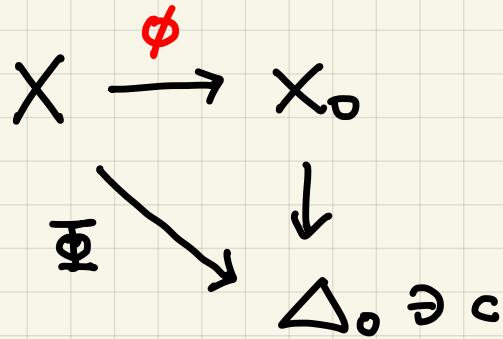
- ϕ is symplecto . on $\overline{\Psi}(\Delta_0 \setminus \{ \bullet \})$
- $\overline{\Psi}(\bullet) \cong S^3$

* In case of (partial) flag varieties with GC-toric deg., every singular fiber of Ψ is a smooth isotropic submfld.

\Rightarrow non-torus lag fibers appears .

Gorenstein

Fano toric degeneration : toric degeneration where X_0 is Gorenstein Fano



Prop. $\Phi(c)$ is monotone in X .

Cor. X should be Fano.

Thm (Nishinou - Nohara - Ueda) If X_0 admits a small resolution ,

$$\text{then } W_{L_c} = \sum_{F: \text{facet}} z^v_F$$

$\tilde{X}_0 \xrightarrow{\sim} X_0$ with exc. locus
has $\text{codim}_{\mathbb{C}} \geq 2$

String polytopes $\Delta_{\mathbf{i}_0}$. Set $G = \mathrm{SL}_{n+1} \mathbb{C}$ and $X = G/B$

- String polytope is a rational convex polytope determined by two

data : $\mathbf{i}_0 = (i_1, \dots, i_N) \in [w_0]$ denoted by $\Delta_{\mathbf{i}_0}(\lambda)$

$\lambda \in \mathbb{Z}_{\geq 0}^n \subseteq \text{wt lattice } \mathbb{Z}\langle \varpi_1, \dots, \varpi_n \rangle$
 ~~~ set of dominant wts

Properties For each  $\lambda \in \mathbb{Z}_{\geq 0}^n$ .

(1)  $|\Delta_{\mathbf{i}_0}(\lambda) \cap \mathbb{Z}^n| = \dim V_\lambda$ ,  $V_\lambda$ : irr. rep. w/ highest wt  $\lambda$ .

(2)  $\lambda : \mathfrak{h}^* \rightarrow \mathbb{Z}$  determines a character  $\lambda : (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$ ,

and it extends to  $\tilde{\lambda} : B \rightarrow \mathbb{C}^* \cong \mathbb{C}^{5B}$

$\Rightarrow \mathcal{L}_\lambda := G \times_B \mathbb{C}$ ,  $(g, z) \cdot b := (gb, \tilde{\lambda}(b)z)$

(3) Let  $X_{\mathbf{i}_0} : \text{toric var. assoc. to } \Delta_{\mathbf{i}_0}$ .

(Alexeev-Briin)  $X_{\mathbf{i}_0}$  is  $\mathbb{Q}$ -Fano. ( $X_0$  is Fano iff  $\Delta_0$  is integral)

Prop  $-K_X \cong \mathcal{L}_{(2,2,\dots,2)}$

(Set  $\Delta_{\mathbf{i}_0} := \Delta_{\mathbf{i}_0}(2,2,\dots,2)$ )

$\Delta_{\mathbf{i}_0}$  contains a unique int. lattice pt.

Gorenstein

Thm (1)  $\exists$  toric degeneration

$$G/B \xrightarrow{\phi_{\alpha}} X_{\alpha}$$

(2) If  $W_{\alpha} : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$  : comb. superpotential , then  
 $W_{\alpha'}$

$W_{\alpha}$  and  $W_{\alpha'}$  are related by "cluster mutation".

\* When  $X_{\alpha}$  admits a small resolution , then

comb. Superpot. = potential ftn