Pontryagin-Thom Construction

Alastair Darby

(Xi'an Jiaotong Liverpool University)

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Unstable Pontryagin Construction

G – compact Lie gp.

Embedding $i : X^n \hookrightarrow M^{n+k}$ smooth, compact

$\circlearrowleft G$ $\quad$ closed $\quad$ $\circlearrowleft G$

normal bundle $\nu = \nu(X \subset M) = i^*TM / TX$

V-framing $f : \nu \xrightarrow{\cong} \mathbb{R}^k \times X$ $\quad$ V G-rep.

Thom space $Tf : T\nu \xrightarrow{\cong} S^k \wedge X_+ = \Sigma^k X_+$

tubular neighbourhood $\varphi : \nu \xrightarrow{\cong} U \subseteq M$

open nbhd of X in M

collapse map $c : M \longrightarrow M/(M \setminus \varphi(\nu)) \underset{(\varphi)}{\cong} T\nu \underset{Tf}{\cong} S^k \wedge X_+ \xrightarrow{pr_1} S^k$

cohomotopy class $[c] \in \pi^k_G(M) = [M, S^k]_G$

$$PT_G : \Omega_{\cancel{n}}^{G}(M) \longrightarrow \pi_{\cancel{n}}^{G}(M)$$

$$X_1^n \sim X_2^n \iff \exists W^{n+1} \subset M \times I \text{ st. } \partial W \subset M \times \{0,1\} \ \& \ \partial W = X_1 \amalg X_2$$

PT is a ~~bijection~~ : given $[f] \in \pi^k(M)$ $\cancel{\exists}$ $g \in [f]$ st. $0 \in S^k$ is a regular value of g.

Take $g^{-1}(0)$.

RESTRICT TO FIXED POINT SUBMANIFOLDS

Partial Framings

Fix G-rep V.

PV-framing $\quad f : \nu \xrightarrow{\text{surj.}} V \times X \iff \nu = \nu_V \oplus \nu_W$

where ν_V is V-framed

<u>NOTE</u> $\dim V \leq k$

Suppose $X = X^G \subset M^G \subset M$

$\Omega^{fix}_{PV}(M)$ bordism classes of PV-framed fixed point submanifolds of M

$X_1^n \sim X_2^n \subset M^G \subset M \iff \exists W^{n+1} \subset M^G \times [0,1]$ st. $\partial W = X_1 \amalg X_2$

Let $X^n \subset M^G$ be a fixed point submanifold

where $n = \dim M^G - \dim V^G$

$\varphi : \nu \xrightarrow{\cong} U \subset M$ G-inva. tub. nbhd. of X^G in M

suppose ν has a PV-framing

$f : \nu \twoheadrightarrow V \times X$

$$M \longrightarrow M/(M \setminus \varphi(\nu)) \cong T\nu \xrightarrow{Tf} S^V \wedge X_+ \xrightarrow{pr_1} S^V$$

hmtpy between G-fixed pt. submflds. can be extended to a G-equiva. hmtpy [Wasserman '69]

So we get a well-defined map

$$PT_{fix} : \Omega^{fix}_{PV}(M) \longrightarrow \pi^{V}_{G}(M)$$

<u>Theorem</u> PT_{fix} is an isomorphism.

<u>Sketch Proof</u> Let $f : M \longrightarrow S^V$ & restrict $f|_{M^G} : M^G \longrightarrow (S^V)^G$

Standard transversality arguments $\Rightarrow \exists\, g \sim f|_{M^G}$ with $0 \in (S^V)^G$ a regular value

Extend this hmtpy to G-equiva hmtpy $H : \tilde{g} \sim f$ st. $\tilde{g}|_{M^G} = g$

Let $X = \tilde{g}^{-1}(0)$ a fixed pt. submfld

Equiv. hmtpy gives hmtpy on M^G by restriction

so $H^{-1}(0) = W \subset M^G \times [0,1]$ gives a fixed pt. bordism between fixed pt. mflds

Fix G-inva. tub nbhd of X $\varphi : \nu \cong U \subset M$

then $f|_\nu : \nu \longrightarrow V \subset S^V$ gives a PV-framing

$\square$

<u>Theorem</u> When G is abelian, $\Omega^{fix}_{PV}(M) = \Omega^G_V(M) \cong \pi^V_G(M)$

[Wasserman '69] or finite.

$X^G \longleftarrow X$

Quotient Orbifolds

Suppose $G \circlearrowright M$ with finite G_x. Then M/G is an orbifold.

Every suborbifold is of the form X/H with $H \leq G$ and X H-inva. submfld.

<u>Lemma.</u>

X_1^n/H bordant to X_2^n/H $\iff$ $\exists$ H-inva. $W^{n+1} \subset M \times I$ st

$$\partial(W/H) = (X_1 \amalg X_2)/H$$

An orbi bundle over M/G is equiv to a G-bun over M

Define a V-framing as a V-framing of the G-bun

$\Omega_V^{orb}(M/G)$ bordism of V-framed suborbifolds of M/G

<u>Theorem</u> $\Omega_V^{orb}(M/G) \cong \bigoplus_{[H] \leq G} \underbrace{\Omega_V^H(M)}_{\text{H-equiv. V-framed submflds of } M}$

where H runs through conjugacy classes of closed subgroups of G

<u>Sketch Proof</u> $\forall H \leq G$, $\exists$ canonical map $\Omega_V^H(M) \longrightarrow \Omega_V^{orb}(M/G)$

which only depends on the conjugacy class of H.

This induces a map $\bigoplus_{[H] \leq G} \Omega_V^H(M) \longrightarrow \Omega_V^{orb}(M/G)$

which is an iso since every suborbifold is a quotient by an H-action for some H. $\square$

<u>Corollary</u> $\Omega_V^{orb}(M/G) \cong \bigoplus_{[H] \leq G} \pi_H^V(M)$

for G abelian or finite.