Pontryagin-Thom Construction

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Unstable Pontryagin Construction

Embedding $i : X^n \hookrightarrow M^{n+k}$ smooth, compact

Normal bundle $\nu = \nu(X \subset M) = i^*TM/\nu X$

Framing $f : \nu \cong V \times V \times V \cong \nu$ $G$-rep.

Thom space $T_f : T\nu \cong S \wedge X_+ = \Sigma X_+$

Tubular neighbourhood $\phi : \nu \cong U \subset M$

Collapse map $c : M \to M/(M, \phi X) \cong T\nu \cong S \wedge X_+ \to S$

Cohomotopy class $[C] \in \pi^V(M) = [M, S]_{Gr}$
\[ \text{PT} : \Omega^G_M \rightarrow \pi^G_M \]

\[ X_1 \sim X_2 \iff \exists W^{n+1} \subset M \times [0,1] \text{ s.t. } \forall W \subset M \times \{0,1\} \text{ and } \forall W = X_1 \sqcup X_2 \]

PT is a bijection: given \([f] \in \pi^k_M\) and \(g \in [f]\) s.t. \(0 \in S^k\) is a regular value of \(g\).

Take \(g^*(0)\).

Restrict to fixed point submanifolds.
Partial Framings

Fix $G$-rep $V$.

$PV$-framing $f : V \xrightarrow{\text{surj.}} V \times X \iff V = V_V \oplus V_W$

where $V_V$ is $V$-framed

**Note** $\dim V \leq k$

Suppose $X = X^G \subset M^G \subset M$

$\Omega^g_{PV}(M)$ bordism classes of $PV$-framed fixed point submanifolds of $M$

$X_1 \sim X_2 \subset M^G \subset M \iff \exists W^{g+1} \subset M^G \times [0,1]$ st. $\partial W = X_1 \sqcup X_2$
Let $X^\circ \subset M^G$ be a fixed point submanifold where $n = \dim M^G - \dim V^G$

$\varphi : \gamma \cong U \subset M$ G-inv. tub. nbhd. of $X^\circ$ in $M$

suppose $\gamma$ has a PV framing

$f: \gamma \rightarrow V \times X$

$M \rightarrow M/(M, \varphi(\nu)) \cong T\gamma \xrightarrow{Tf} S^v \wedge X_+ \xrightarrow{pr_1} S^v$

hfmtpy between G-fixed pt. submflds. can be extended to a G-equiv. hfmtpy [Wasserman '69]
So we get a well-defined map

$$PT_{fix} : \Omega^\text{fix}_{PV}(M) \to \pi_G^V(M)$$

\underline{Theorem} \hspace{1cm} PT_{fix} is an isomorphism.

\underline{Sketch Proof} \hspace{1cm} Let $f : M \to S^v$ & restrict $f|_{M^g} : M^g \to (S^v)^g$

Standard transversality arguments $\Rightarrow$ $\exists g \approx f|_{M^g}$ with $0 \in (S^v)^g$

a regular value

Extend this homotopy to $G$-equivariant homotopy $H : \tilde{g} \approx f$ st. $\tilde{g}|_{M^g} = g$

Let $X = \tilde{g}^{-1}(0)$ a fixed pt. submfd
Equiv. htpy gives htpy on $M^G$ by restriction so $H^*(O) = W \subset M^G \times [0,1]$ gives a fixed pt. bordism between fixed pt. mflds.

Fix $G$-inv. tub nbhd of $X$ $\varphi : V \subseteq U \subseteq M$

then $f|_V : V \to V \subseteq S^v$ gives a $PV$-framing $\Rightarrow$

\begin{align*} \Omega_{PV}^{\text{fix}}(M) = \Omega^a_V(M) \cong \pi^V_{G}(M) \end{align*}

\textbf{Theorem} When $G$ is abelian, $\Omega_{PV}^{\text{fix}}(M) = \Omega^a_V(M) \cong \pi^V_{G}(M)$ or finite.

\textbf{[Wasserman '69]}
Quotient Orbifolds

Suppose $G \times M$ with finite $G_x$. Then $M/G$ is an orbifold. Every suborbifold is of the form $X/H$ with $H \leq G$ and $X$ $H$-inva. submfld.

Lemma.

$X_{i}/H$ bordant to $X_{j}/H$ $\iff$ $\exists$ H-inva. $W$ $^{m}$ $\subset$ $M \times I$ st

$\mathcal{A}(W/H) = (X_{i} \sqcup X_{j})/H$

An orbi-bundle over $M/G$ is equiv to a $G$-bun over $M$

Define a $V$-framing as a $V$-framing of the $G$-bun
\( \Omega^\text{orb}_V(M/G) \) bordism of V-framed suborbifolds of \( M/G \)

**Theorem** \( \Omega^\text{orb}_V(M/G) \cong \bigoplus_{[H] \leq G} \Omega^H_V(M) \mid \text{H-equia. } V\text{-framed submfd of } M \)

where \( H \) runs through conjugacy classes of closed subgroups of \( G \)

**Sketch Proof** \( \forall H \leq G, \exists \) canonical map \( \Omega^H_V(M) \rightarrow \Omega^\text{orb}_V(M/G) \)

which only depends on the conjugacy class of \( H \).

This induces a map \( \bigoplus_{[H] \leq G} \Omega^H_V(M) \rightarrow \Omega^\text{orb}_V(M/G) \)
which is an iso since every suborbifold is a quotient by an $H$-action for some $H$. 

\[ \Omega^\text{orb}(M/G) \cong \bigoplus_{[H] \leq G} \pi^*_H(M) \]

**Corollary** 

For $G$ abelian or finite.