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Equivariant formality of isotropy actions on homogeneous spaces with rank difference one

(Joint with Jeffrey Carlson)

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§1. Introduction

Def. An action of cpt conn. G on a smooth cpt conn. mfd M is Equivariantly formal if the Borel fibration

$$\begin{array}{ccc} M & \longrightarrow & EG \times_G M \\ & & \downarrow \\ & & BG \end{array}$$

is totally non-cohomologous to zero, i.e. the Leray-Serre Spectral sequence collapses at E_2

$$E_2 = E_3 = \cdots = E_\infty .$$

e.g. 1. Manifolds with $H^{odd}(M, \mathbb{Q}) = 0$

A special case: G/H with $rk(G) = rk(H)$

Any action $K \curvearrowright M$ is equivariantly formal.

2. Hamiltonian actions on symplectic manifolds

A special case: toric manifolds

Thm. $T \curvearrowright M$ is equivariantly formal if and only if

$$\dim H^*(M) = \dim H^*(M^T).$$

Non-e.g. 1. $G \curvearrowright M$ with $M^G = \phi$ is Not equivariantly formal

A special case: almost free actions

2. (Shiga 1996)

Consider $S = \left\{ \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} \subset SU(3)$

Then $S \curvearrowright SU(3)/S$ is Not equivariantly formal.

Def. Let G be a cpt conn. Lie group, K be a closed conn. subgroup.

The action of K on G/K by left multiplication will be called isotropy action.

If this action is equivariantly formal, the pair (G, K) is said to be isotropy formal.

Thm. (Shiga & Takahashi 1995, Shiga 1996)

A pair (G, K) is isotropy formal if it satisfies

(P_1) (G, K) is a Cartan pair

i.e., $\dim \mathrm{Im}(H^*(G/K) \to H^*(G)) \cap P_G$

$= rk(G) - rk(H)$

(P_1') The rational homotopy type of G/K is formal in the sense of Sullivan

(P_2) The map $\pi^*: H^*(G/K)^{\pi_0 N_G(K)} \longrightarrow H^*(G)$ is inj

(P_2') The map $\iota^*: H^*(BG) \longrightarrow H^*(BK)^{\pi_0 N_G(K)}$ is surj.

<u>Thm</u>. (Goertches 2012)

The isotropy action on a <u>symmetric space</u> G/K is equivariantly formal.

<u>Thm</u>. (Goertches & Noshari 2016)

The isotropy action on a <u>generalized symmetric space</u> G/K is equivariantly formal.

<u>Thm</u>. (Carlson & Fok 2018)

A pair (G, K) is isotropy formal iff it satisfies (P_1') (P_2')

Carlson 2015, 2019 studied the isotropy formality of (G, S) for S a circular subgroup of G. ($\mathrm{rk}\, S = 1$)

§2. Isotropy formality for corank-one pairs

There are 3 definitions of ranks of G/K

① : $\dim P_{(G,K)} = \dim \mathrm{Im}(H^*(G/K) \to H^*(G)) \cap P_G$

② : $\dim P'_{(G,K)} = \dim \mathrm{Ker}(P_G \longrightarrow P_K)$ Onishchik 1963

② : $\dim P'_{(G,K)} = \dim \mathrm{Ker}(P_G \longrightarrow P_K)$ Onishchik 1963

③ : $\dim P_G - \dim P_K = \mathrm{rk}\, G - \mathrm{rk}\, K$ Hsiang & Su 1968

$$①, ③ \leq ②$$

<u>Def.</u> Given a pair (G, K), the rank difference $\mathrm{rk}\,G - \mathrm{rk}\,K$ will be called the corank.

<u>Prop.</u> (Carlson 2015, 2019)

A pair (G,K) is isotropy formal iff (G,S) is so, where S is a maximal torus of K.

From now on, assume the corank $\mathrm{rk}\,G - \mathrm{rk}\,K = 1$.

$$\begin{array}{ccc} K & \subset & G \\ \cup & & \cup \\ S & \subset & T \end{array} \qquad \dim T - \dim S = 1$$

$\mathfrak{s} \hookrightarrow \mathfrak{t}$ is a hyperplane,

$\exists\, \alpha \in \mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ s.t. $\mathfrak{s} = \ker(\alpha)$

Consider the coadjoint action $G \circlearrowright \mathfrak{g}^*$ and the coadjoint orbit $\mathcal{O}_\alpha \cong G/G_\alpha$.

Equivalently, $\exists\, v \in \mathfrak{t}_{\mathbb{Z}}$ s.t. $\alpha = (v, \ )$

$$\mathbb{R}v \perp \mathfrak{s}, \qquad S^\perp \triangleq \exp(\mathbb{R}v)$$

$$T = S \times S^\perp.$$

<u>Lem.</u> i) We have $G_\alpha = Z_G(S^\perp)$,

$$\mathcal{O}_\alpha \cong G/G_\alpha = G/Z_G(S^\perp) \cong \mathcal{O}_v .$$

ii) G_α contains T with roots

$$\Phi_\alpha = \{ \beta \in \Phi \mid (\beta,\alpha)=0 \}.$$

$$\mathrm{Span}_{\mathbb{R}}\{ h_\beta \mid \beta\in\Phi_\alpha\} \subseteq \ker\alpha = s .$$

Write $G_\alpha = [G_\alpha, G_\alpha]\cdot T$

Denote $H_\alpha \triangleq [G_\alpha, G_\alpha]\cdot S$, $N \triangleq \pi_0 N_G(S) = N_G(S)/Z_G(S)$

Lem. i) $W_{H_\alpha} \cong W_{G_\alpha} \cong W_\alpha = \{ w\in W_G \mid w\alpha=\alpha\}$ $\qquad W \circlearrowright \mathfrak{t}_{\mathbb{Z}}^* \ni \alpha$

ii) Both $N, W_{H_\alpha} \subset GL(s)$, and

Regarding P_2'

$$N \cong \begin{cases} \langle W_{H_\alpha}, w_0|_s \rangle & \text{if } w_0\alpha=-\alpha \text{ and } w_0|_s \notin W_{H_\alpha} \\ W_{H_\alpha} & \text{otherwise.} \end{cases}$$

Lem. The Euler classes of the circle bundles

$$\begin{tikzcd} S^\perp \cong T/S \arrow[r, "\cong"] \arrow[d] & G_\alpha/H_\alpha \arrow[d] \\ G/S \arrow[r] \arrow[d] & G/H_\alpha \arrow[d] \\ G/T \arrow[r] & G/G_\alpha \end{tikzcd}$$

are both identified as

$\alpha \in H^2(G/G_\alpha), H^2(G/T)$.

Prop. $$H^{\text{even}}(G/H_\alpha) \cong \frac{H^*(BH_\alpha)}{\langle \mathrm{Im}(H^*(BG)\to H^*(BH_\alpha))^+ \rangle}.$$

$$H \quad (G/H_\alpha) \cong \frac{\quad}{\langle \mathrm{Im}(H^*(BG)\to H^*(BH_\alpha))^+\rangle}.$$

Regarding P_1

<u>Thm.</u> (c.f. Greub, Halperin & Vanstone 1976, Onishchik 1994)

If (G,H) is a Cartan pair, then

$$H^*(G/H) \cong \frac{H^*(BH)}{\langle \mathrm{Im}(H^*(BG)\to H^*(BH))^+\rangle} \otimes \Lambda P_{(G,H)}$$

<u>Thm.</u> (Carlson-H.)

A corank-one pair (G,K) is isotropy formal iff either one of the following two cases holds

i) $H^*(G/H_\alpha) \cong H^*(S^{odd})$ and $N = W_\alpha$,

ii) $H^*(G/H_\alpha) \cong H^*(S^{even} \times S^{odd})$ and $N \neq W_\alpha$.

§3. The classification of G/H with $H^*(G/H) \cong H^*(S^{even} \times S^{odd})$

Without loss of generality, assume $\pi_1(G/H) = 1$ or $\pi_1(G/H) = \mathbb{Z}$

<u>Def.</u> A transitive action $G \circlearrowright M$ is irreducible if no proper, normal subgroup of G is transitive on M.

The split case: $G/H \cong G_1/H_1 \times G_2/H_2$.

Table 3.17: Even-dimensional simply-connected homogeneous rational cohomology spheres with irreducible, almost effective G-action Montgomery & Samelson, Borel

$H^*(S^{\text{even}})$

G	H	$\dim G/H$	G/H
$\mathrm{SO}(2k+1)$	$\mathrm{SO}(2k)$	$2k$	S^{2k}
G_2	$\mathrm{SU}(3)$	6	S^6

Table 3.18: Odd-dimensional simply-connected homogeneous rational cohomology spheres with irreducible, almost effective G-action [Oni63, Table 2, p. 457] [Oni94, Table 10, p. 265] [Bes78, Thm. F.7.50-7.54, p. 195-196] [Bes87, Ex. B.7.13, p. 179][Kra02, p. 64-66][KZ04, Table III, p. 154]

$H^*(S^{\text{odd}})$

G		H	$Z_G(H)^0$	$\dim G/H$	G/H
$\mathrm{SU}(k+1)$,	$k \geqslant 2$	$\mathrm{SU}(k)$	$\mathrm{U}(1)$	$2k+1$	S^{2k+1}
$\mathrm{SU}(3)$		$\mathrm{SO}(3)_4$	1	5	
$\mathrm{SO}(2k+1)$,	$k \geqslant 2$	$\mathrm{SO}(2k-1)$	$\mathrm{SO}(2)$	$4k-1$	$V_2(\mathbb{R}^{2k+1})$
$\mathrm{Spin}(9)$		$\mathrm{Spin}(7)$	1	15	S^{15}
$\mathrm{Spin}(7)$		G_2	1	7	S^7
$\mathrm{Sp}(k)$,	$k \geqslant 1$	$\mathrm{Sp}(k-1)$	$\mathrm{Sp}(1)$	$4k-1$	S^{4k-1}
$\mathrm{Sp}(2)$		$\mathrm{SU}(2)_{10} : {}^{\mathbb{H}}\rho_{3\lambda_1}$	1	7	Berger 7-space
$\mathrm{SO}(2k)$,	$k \geqslant 3$	$\mathrm{SO}(2k-1)$	1	$2k-1$	S^{2k-1}
G_2		$\mathrm{SU}(2) : {}^{\mathbb{R}}\rho_{\lambda_1}$	$\mathrm{SU}(2)_3$	11	$V_2(\mathbb{R}^7)$
G_2		$\mathrm{SU}(2)_3 : {}^{\mathbb{R}}\rho_{\lambda_1} + {}^{\mathbb{R}}\rho_{2\lambda_1}$	$\mathrm{SU}(2)$	11	
G_2		$\mathrm{SO}(3)_4 : 2 \cdot {}^{\mathbb{R}}\rho_{2\lambda_1}$	1	11	
G_2		$\mathrm{SO}(3)_{28} : {}^{\mathbb{R}}\rho_{6\lambda_1}$	1	11	

Classification of $G/H \overset{\text{homeo}}{\simeq} S^{\text{even}} \times S^{\text{odd}}$: Kamerich 1977

Classification of $G/H \overset{\text{rational}}{\simeq} S^{\text{even}} \times S^{\text{odd}}$:

Kramer 2002 for odd > even ≥ 4

Wolfrom 2002 for odd > even = 2

Bletz-Siebert 2002 for even > odd = 1

Prop. (Carlson-H.)

If a 1-conn. $G/H \overset{\text{rational}}{\simeq} S^{\text{even}} \times S^{\text{odd}}$ with even > odd ≥ 3

If a 1-conn. $G/H \overset{\text{rational}}{\simeq} S^{\text{even}} \times S^{\text{odd}}$ with $\text{even} > \text{odd} \geq 3$

then $G/H \cong G_1/H_1 \times G_2/H_2$.

<u>Prop.</u> (Carlson-H.)

If a 1-conn. $G/H \overset{\text{rational}}{\simeq} S^{\text{even}} \times S^{\text{even}'}$

then $G/H \cong G_1/H_1 \times G_2/H_2$.

Table 3.24: Simply-connected homogeneous spaces of the rational homotopy type of $S^n \times S^m$ $(n, m \geqslant 2)$ with irreducible, almost effective G-action

G	H	(m,n)	G/H
$\mathrm{SU}(3)$	$i_{p,q}\mathrm{U}(1)$	$(2,5)$	
$\mathrm{Sp}(2)$	$i_{p,q}\mathrm{U}(1)$	$(2,7)$	
G_2	$i_{p,q}\mathrm{U}(1)$	$(2,11)$	
$K_1 \times \mathrm{Sp}(1)$	$H_1 \cdot i_{p,q}\mathrm{U}(1)$, $\quad p \cdot q \neq 0$	$(2,n)$	
$\mathrm{SU}(5)$	$\mathrm{SU}(2) \oplus \mathrm{SU}(3)$	$(4,9)$	$\widetilde{G}_2(\mathbb{C}^5)$
$\mathrm{SU}(4)$	$\mathrm{SU}(2) \oplus \mathrm{SU}(2)$	$(4,5)$	$\widetilde{G}_2(\mathbb{C}^4) \cong V_2(\mathbb{R}^6)$
$\mathrm{SU}(4)$	$\mathrm{SO}(4)$	$(4,5)$	$\widetilde{G}_3(\mathbb{R}^6)$
$\mathrm{SO}(2k+1)$, $\quad k \geqslant 3$	$\mathrm{SO}(2k-2)$	$(2k-2, 4k-1)$	$V_3(\mathbb{R}^{2k+1})$
$\mathrm{Spin}(9)$	$\mathrm{SU}(4)$	$(6,15)$	
$\mathrm{Spin}(7)$	$\mathrm{SU}(3)$	$(6,7)$	$V_2(\mathbb{R}^8) = S^6 \times S^7$
$\mathrm{SO}(7)$	$\mathrm{SO}(3) \oplus \mathrm{SO}(3) \oplus [1]$	$(4,11)$	
$\mathrm{SO}(7)$	$\mathrm{SO}(3) \oplus \mathrm{SU}(2)$	$(4,11)$	
$\mathrm{Sp}(4)$	$\mathrm{SU}(4)$	$(6,15)$	
$\mathrm{Sp}(3)$	$\mathrm{SU}(3)$	$(6,7)$	
$\mathrm{Sp}(3)$	$\mathrm{Sp}(1) \oplus \mathrm{Sp}(1) \oplus [1]$	$(4,11)$	
$\mathrm{Sp}(3)$	$\mathrm{Sp}(1) \oplus \mathrm{SU}(2)_{10}$	$(4,11)$	
$\mathrm{Sp}(3)$	$\mathrm{Sp}(1) \oplus \Delta_2\mathrm{Sp}(1)$	$(4,11)$	
$\mathrm{Sp}(3)$	$\mathrm{SO}(3) \cdot \Delta_3\mathrm{Sp}(1)$	$(4,11)$	
$\mathrm{SO}(2k)$, $\quad k \geqslant 4$	$\mathrm{SO}(2k-2)$	$(2k-2, 2k-1)$	$V_2(\mathbb{R}^{2k})$
$\mathrm{Spin}(10)$	$\mathrm{SU}(5)$	$(6,15)$	
F_4	$\mathrm{Spin}(7)$	$(8,23)$	
F_4	$\mathrm{Sp}(3)$	$(8,23)$	
$\mathrm{Sp}(k) \times \mathrm{Sp}(2)$, $\quad k \geqslant 2$	$\mathrm{Sp}(k-1) \times \Delta\mathrm{Sp}(1) \times \mathrm{Sp}(1)$	$(4, 4k-1)$	
$G_2 \times \mathrm{Sp}(2)$	$\mathrm{Sp}(1) \cdot \Delta\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$	$(4,11)$	
$K_1 \times K_2$	$H_1 \times H_2$	(m,n)	$K_1/H_1 \times K_2/H_2$

§4. Case analysis of isotropy formality

Thm. (Carlson - H.)

With respect to the above table, the following pairs are Not isotropy formal:

- $(\mathrm{SU}(3), i_{p,q}\mathrm{U}(1))$ *with* $(p,q) \in \mathbb{Z}^2 \setminus \{(1,0),(0,1),(1,-1)\}$ *a pair of coprime integers,*
- $(\mathrm{SU}(k+1) \times \mathrm{Sp}(1), \mathrm{SU}(k) \cdot i_{p,q}\mathrm{U}(1))$, *where* $k \geqslant 2$ *and* $p \cdot q \neq 0$,
- $(\mathrm{SU}(5), \mathrm{SU}(2) \oplus \mathrm{SU}(3))$,
- $(\mathrm{Sp}(3), \mathrm{Sp}(1) \oplus \mathrm{SU}(2)_{10})$, $(\mathrm{Sp}(3), \mathrm{SO}(3) \cdot \Delta_3\mathrm{Sp}(1))$,
- $(\mathrm{Spin}(10), \mathrm{SU}(5))$,
- $(F_4, \mathrm{Spin}(7))$, $(F_4, \mathrm{Sp}(3))$,
- $(\mathrm{Sp}(k) \times \mathrm{Sp}(2), \mathrm{Sp}(k-1) \cdot \Delta\mathrm{Sp}(1) \cdot \mathrm{Sp}(1))$ *with* $k \geqslant 2$, $(G_2 \times \mathrm{Sp}(2), \mathrm{Sp}(1) \cdot \Delta\mathrm{Sp}(1) \cdot \mathrm{Sp}(1))$.