

Hessenberg varieties and toric orbifolds

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joint works with

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Introduction

$\text{Hess}(X, h) \subset F\ell(\mathbb{C}^n)$ Hessenberg variety

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$$H^*(\text{Hess}(N, h))$$

logarithmic derivation module
in hyperplane arrangement

$$S_n \curvearrowright H^*(\text{Hess}(S, h))$$

Stanley's chromatic symmetric func.
in graph theory

Hessenberg function

Definition of Hessenberg function

$$h : [n] \rightarrow [n] \text{ **Hessenberg ft.** } \iff \begin{array}{l} h(1) \leq h(2) \leq \dots \leq h(n) \\ h(i) \geq i \quad (i = 1, 2, \dots, n) \end{array}$$

($[n] := \{1, 2, \dots, n\}$)

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Ex. $n = 5$. $h = (3, 3, 4, 5, 5)$ is a Hessenberg ft.

Hessenberg function

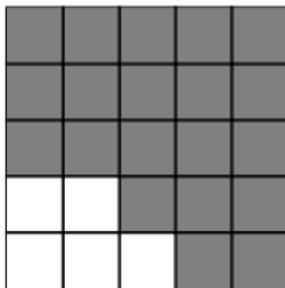
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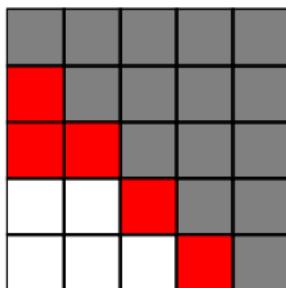
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Hessenberg variety

$$\mathcal{F}\ell(\mathbb{C}^n) := \{(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i, 1 \leq i \leq n\}$$

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We consider $\text{Hess}(N, h)$ and $\text{Hess}(S, h)$.

$$N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad S = \begin{pmatrix} c_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & c_n \end{pmatrix} \quad (c_i \neq c_j).$$

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Red				
Red	Red			
		Red		
				Red

$$\dim_{\mathbb{C}} \text{Hess}(N, h) = \dim_{\mathbb{C}} \text{Hess}(S, h) = 5$$

Hess(N, h) v.s. Hess(S, h)

Theorem [H.Abe-Harada-H-Masuda]

The following ring isomorphism holds:

$$H^*(\text{Hess}(N, h); \mathbb{Q}) \cong H^*(\text{Hess}(S, h); \mathbb{Q})^{S_n}$$

where $S_n \curvearrowright H^*(\text{Hess}(S, h); \mathbb{Q})$ is Tymoczko's dot action.

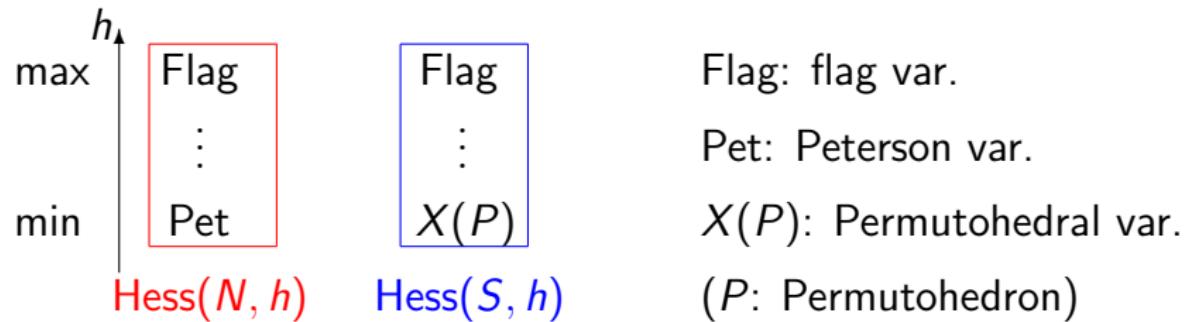
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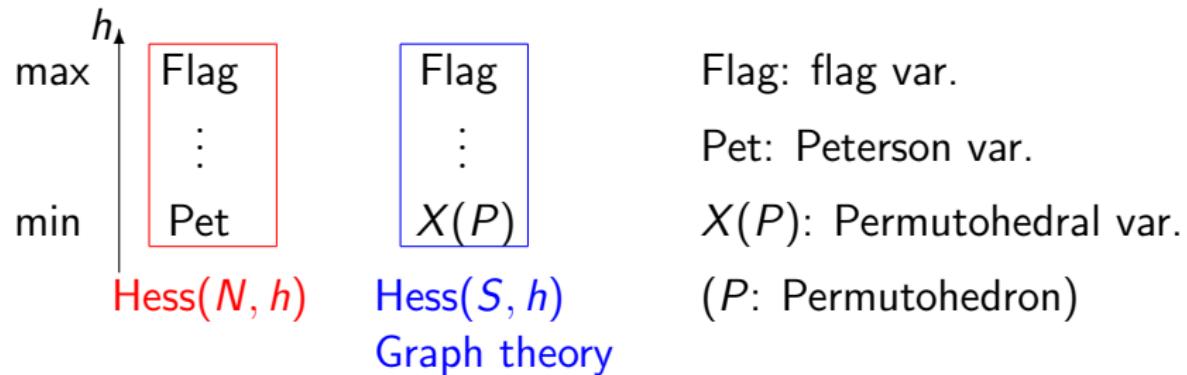
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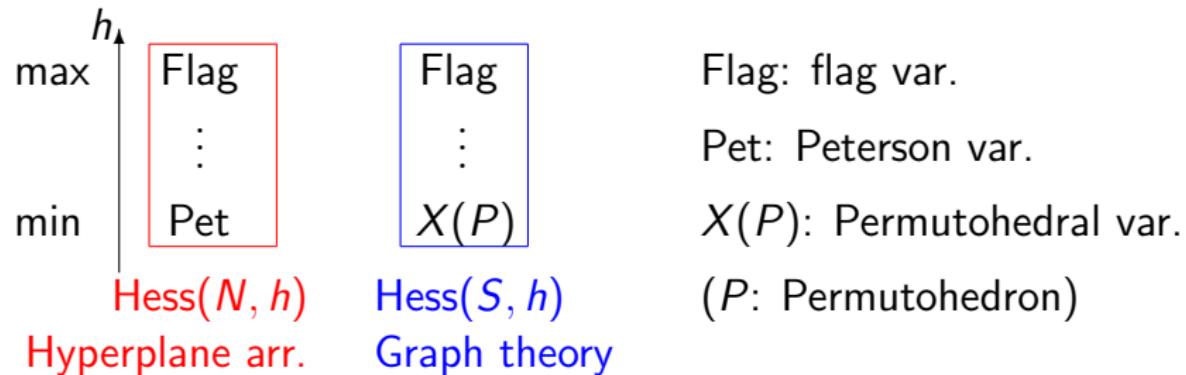
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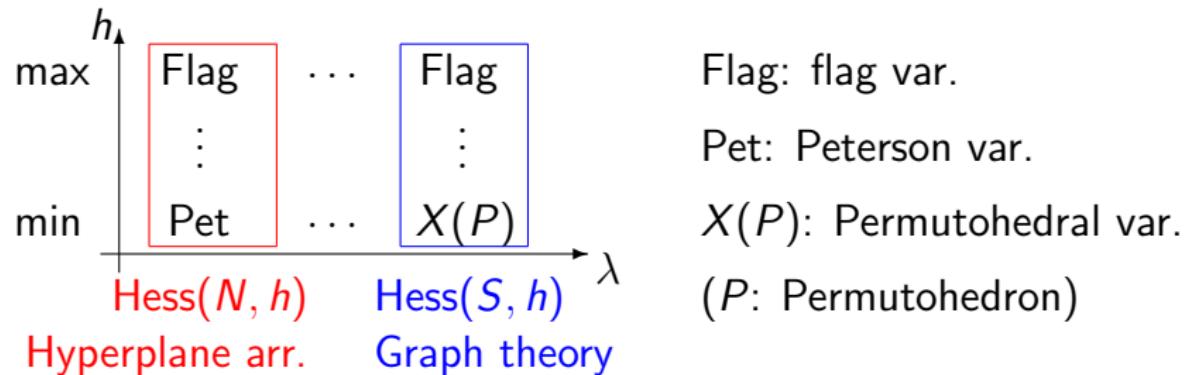
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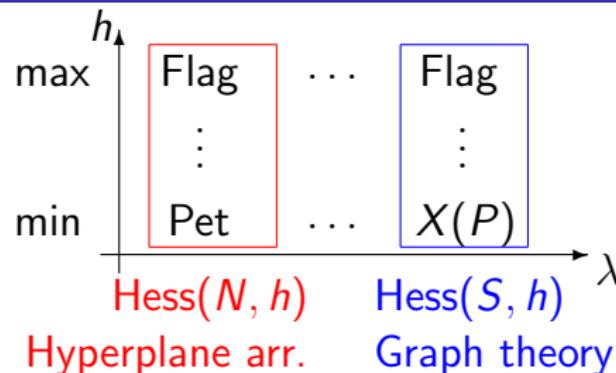
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Regular Hessenberg varieties



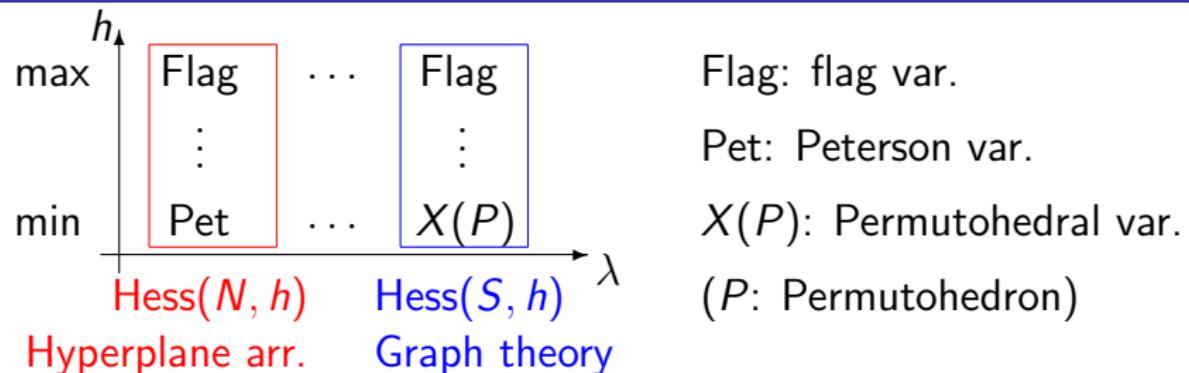
Flag: flag var.

Pet: Peterson var.

$X(P)$: Permutohedral var.

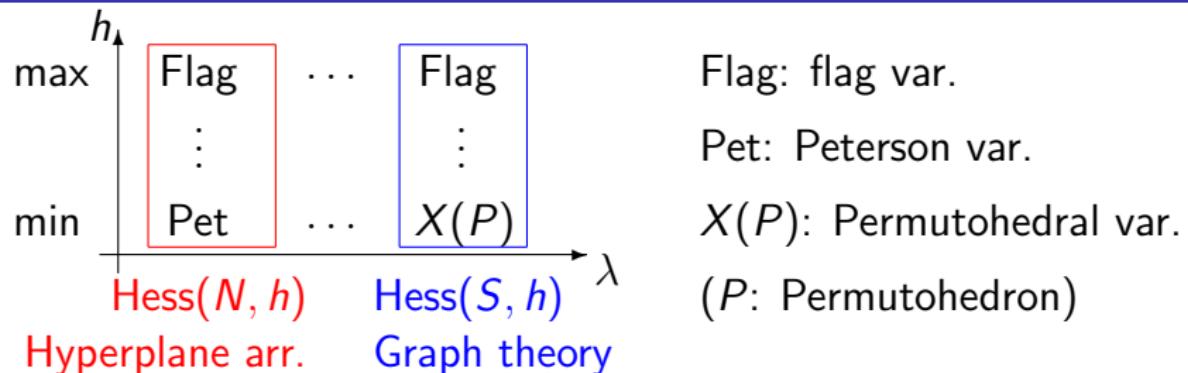
(P : Permutohedron)

Regular Hessenberg varieties



Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of n .

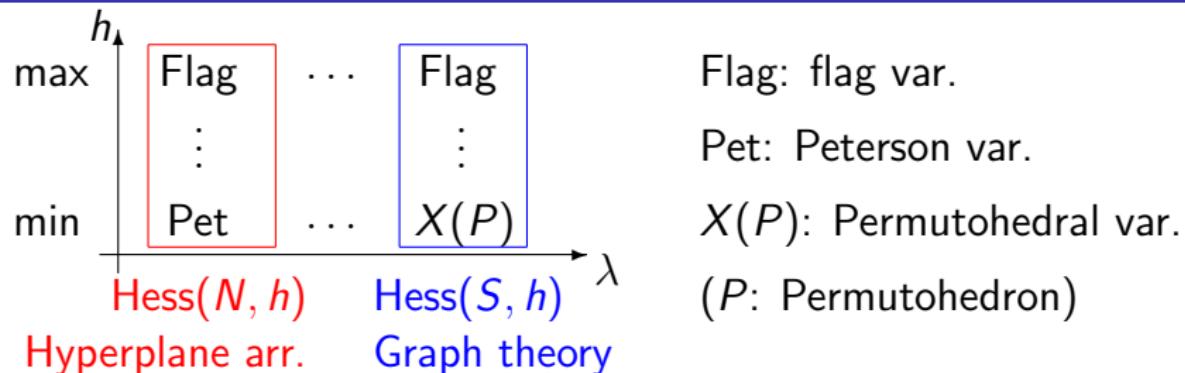
Regular Hessenberg varieties



Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of n .

$$R_\lambda = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_\ell \end{pmatrix}, \quad J_i = \begin{pmatrix} c_i & 1 & & \\ & \ddots & \ddots & \\ & & c_i & 1 \\ & & & c_i \end{pmatrix} \quad (c_i \neq c_j).$$

Regular Hessenberg varieties



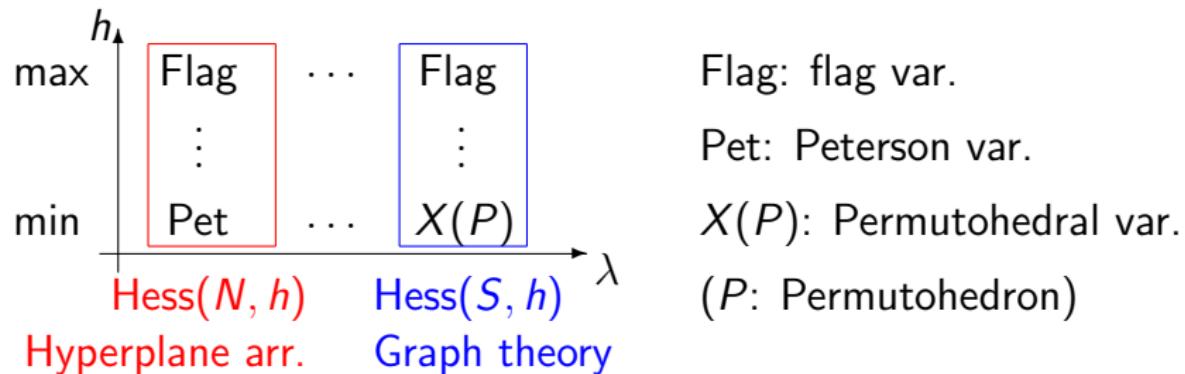
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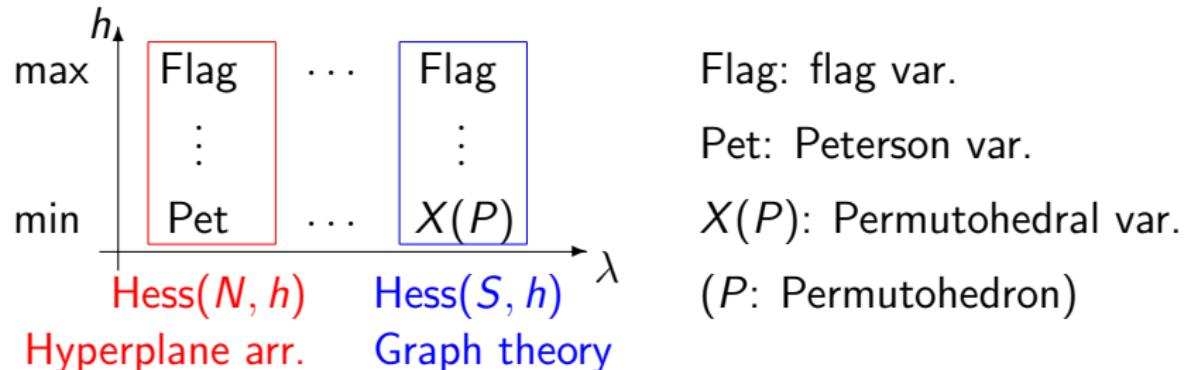
If $\lambda = (n) \Rightarrow \text{Hess}(R_\lambda, h) = \text{Hess}(N, h)$

If $\lambda = (1, \dots, 1) \Rightarrow \text{Hess}(R_\lambda, h) = \text{Hess}(S, h)$

Invariants



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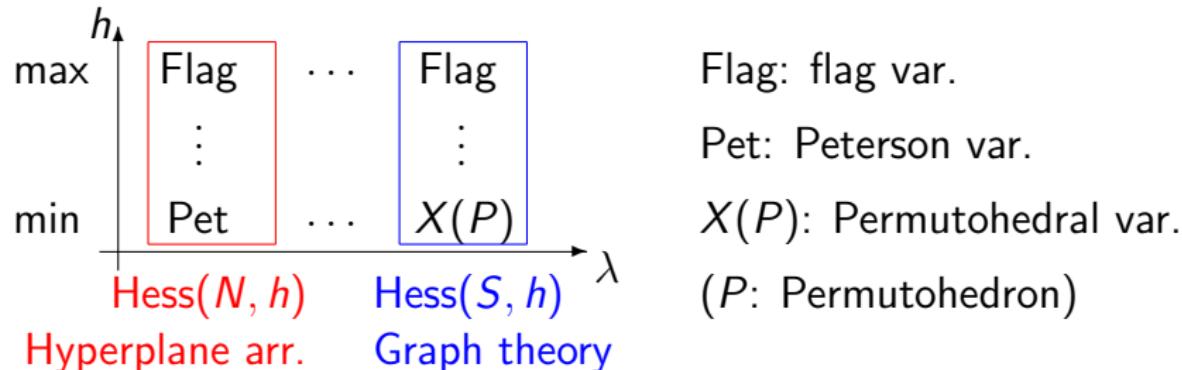


Theorem [Brosnan-Chow, Balibanu-Crooks, Vilonen-Xue]

$$H^*(\text{Hess}(R_\lambda, h); \mathbb{Q}) \cong H^*(\text{Hess}(S, h); \mathbb{Q})^{S_\lambda}$$

Here, $S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}$.

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Theorem [H-Masuda-Shareshian-Song]

$$H^*(X(P); \mathbb{Q})^{S_\lambda} \cong H^*(X(P/S_\lambda); \mathbb{Q})$$

Hessenberg varieties and Toric orbifolds

Corollary

$$H^*(\text{Hess}(R_\lambda, h_{\min}); \mathbb{Q}) \cong H^*(X(P); \mathbb{Q})^{S_\lambda} \cong H^*(X(P/S_\lambda); \mathbb{Q})$$

Hessenberg varieties and Toric orbifolds

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