The Gromov width of symplectic toric manifolds associated with graphs

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Hamiltonian torus action

- (M^{2n}, ω): symplectic manifold of dimension $2n$
- $T^k \cong (\mathbb{R}/\mathbb{Z})^k$: k-dimensional torus acting on M
- $\mathfrak{t} \cong \mathbb{R}^k$: Lie algebra of T^k with dual Lie algebra $\mathfrak{t}^*$
- $\underline{X}$: fundamental vector field on M for given $X \in \mathfrak{t}$

Definition

T^k-action on (M, ω) is called Hamiltonian if for each $X \in \mathfrak{t}$ there exists $\mu\colon M \to \mathfrak{t}^*$ such that

$$\omega(-, \underline{X}) = d\langle \mu, X \rangle.$$

Such μ is called a moment map.

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Fix basis $X_1, \ldots, X_k$ of $\mathfrak{t}_{\mathbb{Z}}$. This identifies $(\mathfrak{t}^*, \mathfrak{t}^*_{\mathbb{Z}})$ with $(\mathbb{R}^n, \mathbb{Z}^n)$.

Any quantity should be measured with respect to the lattice.

Examples of moment maps

1. T^2 acts on $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ by $(t_1, t_2) \cdot (z_1, z_2) := (e^{2\pi i t_1} z_1, e^{2\pi i t_2} z_2)$.

$$\mu(z_1, z_2) = (\pi |z_1|^2, \pi |z_2|^2).$$

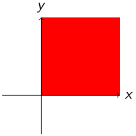
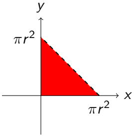


image is the first quadrant



when restricted to open ball $B^4(r)$

2. σ: volume form on S^2 so that $\int_{S^2} \sigma = 1$.
T^2 acts on $(S^2 \times S^2, a\sigma + b\sigma)$ diagonally.

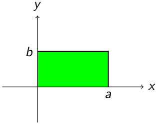
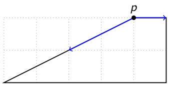


image is a polytope

Delzant theorem

- A symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian T^n-action is called toric.
- In toric case, the image of a moment map $\mu \colon M \to \mathbb{R}^n$ is a polytope satisfying the following conditions:

1. There are n edges at each vertex p.
2. The edges at p has the form $p + tu_i$ for some basis vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$.



Such polytopes are called Delzant polytopes.

Theorem (Delzant)

There is a 1-1 correspondence

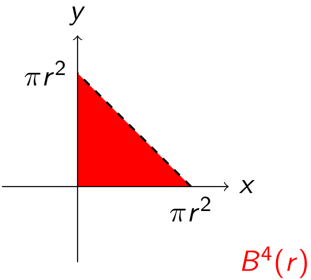
$$\{\textit{closed symplectic toric manifolds}\} \leftrightarrow \{\textit{Delzant polytopes}\}.$$

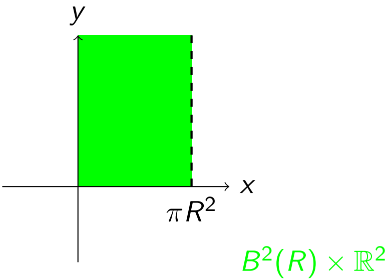
Want: recover geometric data from Delzant polytopes

Gromov width

Theorem (Gromov nonsqueezing)

$B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ *symplectically if and only if* $r \leq R$.

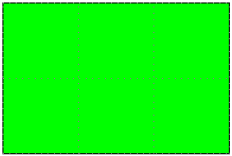




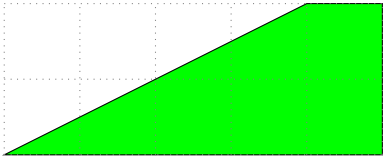
(Caution: this picture is far from a proof.)

Definition (Gromov width)

$w_G(M^{2n}, \omega) := \sup\{\pi r^2 \mid B^{2n}(r) \hookrightarrow (M, \omega) \text{ symplectically}\}$.



$S^2 \times S^2$



Hirzebruch surface $\mathcal{H}_2$

These two are different as symplectic toric manifolds,
but they are isomorphic as symplectic manifolds.

- Delzant polytope = symplectic structure + torus action
  $\Longrightarrow$ symplectic data should be independent of the action.
- In these examples, the Gromov width is the height.

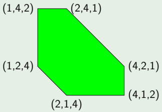
Nestohedron

- A building set $\mathcal{B}$ on $[n+1]$ is a collection of subsets in $[n+1]$ satisfying the following conditions:
  1. Each singleton is an element of $\mathcal{B}$.
  2. $I, J \in \mathcal{B}$ with $I \cap J \neq \emptyset \implies I \cup J \in \mathcal{B}$.
- For $I \in \mathcal{B}$, let $\Delta_I := conv\{e_i \mid i \in I\}$.
- The nestohedron $P_{\mathcal{B}}$ is defined to be the Minkowski sum

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I.$$

Example ($n = 2, \mathcal{B} = \{1, 2, 3, 12, 13, 23, 123\}$)

$$\begin{aligned} P_{\mathcal{B}} &= e_1 + e_2 + e_3 \\ &\quad + conv\{e_1, e_2\} + conv\{e_1, e_3\} + conv\{e_2, e_3\} \\ &\quad + conv\{e_1, e_2, e_3\} \end{aligned}$$



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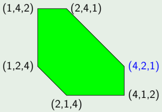
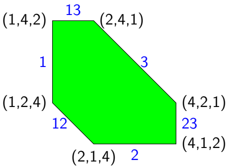


Properties

- $P_{\mathcal{B}} \subset \{\sum_{i=1}^{n+1} x_i = |\mathcal{B}|\}$.
- $P_{\mathcal{B}}$ is a Delzant polytope.
- If $[n+1] \in \mathcal{B}$, then $P_{\mathcal{B}}$ is n-dimensional.
  ($[n+1] \notin \mathcal{B}$, then $P_{\mathcal{B}}$ is a product of smaller nestohedra.)
- $I \subsetneq [n+1]$ defines a facet of $P_{\mathcal{B}}$:

$$F_I \subset \left\{ \sum_{i \in I} x_i = |\mathcal{B}|_I| \right\}, \quad \text{where } \mathcal{B}|_I = \{J \in \mathcal{B} \mid J \subset I\}.$$

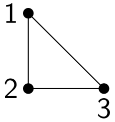


Graph associahedron

Building set from a simple graph

- G: simple graph with vertex set $[n+1]$.
- $\mathcal{B}(G) := \{I \subset [n+1] \mid$ the subgraph $G|_I$ induced by I is connected$\}$.

$P_{\mathcal{B}(G)}$ is called a graph associahedron.

$G =$ 

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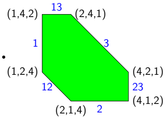
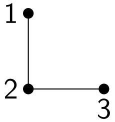


$G =$ 

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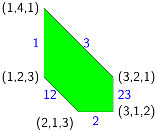


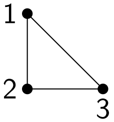
Main theorem

Theorem

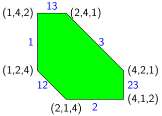
Let G be a simple graph with the vertex set $[n+1]$.
The Gromov width of the symplectic toric manifold for $P_{\mathcal{B}(G)}$ is

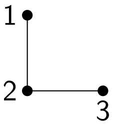
$$\min\{k_i > 1 \mid i = 1, \ldots, n+1\} - 1,$$

where k_i is the number of connected induced subgraphs of G containing i.

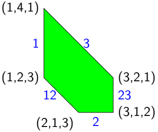
$G =$ 

$$k_1 = k_2 = k_3 = 4.$$
$$\Longrightarrow \quad w_G = 3.$$



$G =$ 

$$k_1 = k_3 = 3, \; k_2 = 4.$$
$$\Longrightarrow \quad w_G = 2.$$



For $I \in \mathcal{B}$, F_I has a parallel facet if and only if $[n+1] \setminus I \in \mathcal{B}$.

When $\mathcal{B}$ is obtained from a graph,

1. Such $I \in \mathcal{B}$ always exists.
2. Minimal distance between such facets are attained when I or its complement is a singleton.
3. $k_i - 1$ is the distance between $F_{\{i\}}$ and $F_{[n+1]\setminus\{i\}}$.

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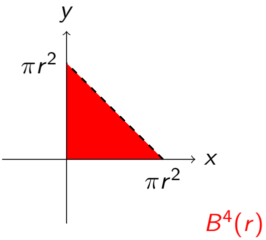

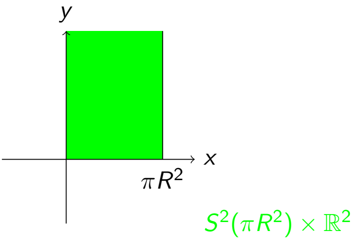

Distance between parallel facets bounds the Gromov width.

Stabilized embedding

- Gromov nonsqueezing: $w_G(S^2(\pi r^2) \times \mathbb{R}^2) = \pi r^2$.
- On the other hand, $w_G(\Sigma_g(\pi r^2) \times \mathbb{R}^2) = \infty$ for $g \geq 1$.

Let M_1, M_2 be symplectic toric manifolds. Is it true that

$$w_G(M_1 \times M_2) = \min\{w_G(M_1), w_G(M_2)\}?$$

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For M_G constructed from graph G, $w_G(M_G \times \mathbb{R}^2) = w_G(M_G)$.

Corollary

Let H be a subgraph of G. Suppose $k = |G| - |H| > 0$. Then

$$M_G \times \mathbb{R}^{2m} \hookrightarrow M_H \times \mathbb{R}^{2k+2m}$$

can never be symplectic for any $m \geq 0$.

Is there a topological obstruction to this embedding?

Idea of proof

Theorem

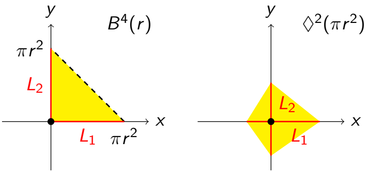
Let G be a simple graph with the vertex set $[n+1]$.
The Gromov width of the symplectic toric manifold for $P_{\mathcal{B}(G)}$ is

$$\lambda := \min \{k_i > 1 \mid i = 1, \ldots, n+1\} - 1,$$

where k_i is the number of connected induced subgraphs of G containing i.

- (Lower bound) To show $w_G \geq \lambda$,
  1. Use global action-angle coordinates given by moment map.
  2. Find some shape (corresponding to a ball) of "size" λ inside P.
- (Upper bound) To show $w_G \leq \lambda$,
  1. Find J-holomorphic sphere with symplectic area $\leq \lambda$.
  2. Use McDuff–Tolman computation on Seidel representation to find suitable nonvanishing Gromov–Witten invariant.
  3. Use semifree circle action with codimension 2 extrema.

Lower bound



Find $L_1, \dots, L_n$ satisfying

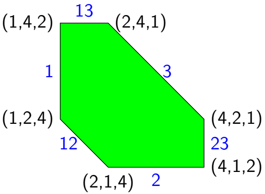
- $L_1 \cap \cdots \cap L_n$ is a point.
- Primitive vectors parallel to L_i form a basis for $\mathbb{Z}^n$.
- L_i has affine length ρ.

$$\Diamond^n(\rho) := conv(L_1, \dots, L_n).$$

Let P be the moment map image of a symplectic toric manifold.

Theorem (Mandini–Pabiniak, Latschev–McDuff–Schlenk)

$\Diamond^n(\rho) \subset P \implies w_G \geq \rho.$



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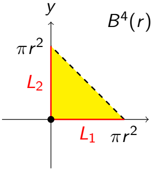

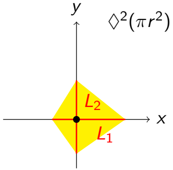

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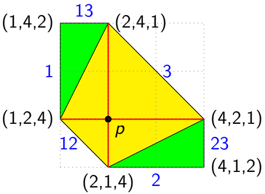

- $w_G \geq 3$.
- $p = (2, 2)$
  if 3rd coordinate is ignored.

Finding $\Diamond$ inside P

- G: connected simple graph with the vertex set $[n+1]$.
- $\mathcal{B}$: building set constructed from G.
- k_i: number of connected induced subgraphs of G containing i.

Assume k_{n+1} is minimal among k_i.
Regard $P_{\mathcal{B}}$ as a subset in $\mathbb{R}^n$ by forgetting last coordinate.

Goal: find $\Diamond^n(k_{n+1}-1)$ in $P_{\mathcal{B}}$

Take $L_i = \{(a, \ldots, a, x, a, \ldots, a) \mid 1 \leq x \leq k_{n+1}\}$, where

$$a := \frac{|\mathcal{B}| - k_{n+1} - 1}{n-1}.$$

Checking $L_i \subset P_{\mathcal{B}}$ and $1 \leq a \leq k_{n+1}$ reduces to the following.

Lemma

$$n \cdot k_i \geq |\mathcal{B}| - 1 \quad \textit{for any } i = 1, \ldots, n+1.$$

Gromov–Witten invariants

(Upper bound) To show $w_G \leq \lambda$,

1. Find J-holomorphic sphere with symplectic area $\leq \lambda$.
2. Use Seidel representation to find nonzero GW-invariant.
3. Use semifree circle action with codimension 2 extrema.

The (genus zero) Gromov–Witten invariant

$$\mathrm{GW}^M_{A,k}(\alpha_1, \dots, \alpha_k) \in \mathbb{Q}$$

counts number of J-holomorphic spheres in class $A \in H_2(M, \mathbb{Z})$, passing through cycles $\alpha_i \in H^*(M)$.

Theorem (Gromov)

If $\mathrm{GW}^M_{A,k}([pt], \alpha_2, \dots, \alpha_k) \neq 0$ for some $A \in H_2(M, \mathbb{Z})$, $\alpha_i \in H^*(M)$, then the Gromov width of (M, ω) is at most $\omega(A) > 0$.

Seidel representation

Goal: find A, α_i such that $\mathrm{GW}^M_{A,k}([pt], \ldots, \alpha_k) \neq 0$, $\omega(A) = \lambda$.

- The Seidel morphism is a group homomorphism

$$S : \pi_1(\mathrm{Ham}(M, \omega)) \to (QH^0(M; \Lambda)^\times, *).$$

- $QH^\bullet(M; \Lambda) = H^\bullet(M) \otimes \Lambda$ with quantum product $*$.
- $\Lambda = \{\sum a_i q^{\mu_i} t^{\kappa_i} \mid \deg q = 2, \deg t = 0,$ some condition$\}$.
- $a * b = \sum_{A \in H_2(M,\mathbb{Z})} (a * b)_A \otimes q^{c_1(A)} t^{\omega(A)}$, where for all c,

$$\int_M (a * b)_A \cup c = \mathrm{GW}^M_{A,3}(a, b, c).$$

We obtain information on GW invariants by studying S.

Upper bound

- $u \in \pi_1(\mathrm{Ham}(M,\omega))$ is represented by Hamiltonian S^1-action.
  $\implies -u$ is represented by the opposite S^1-action.
- $S : \pi_1(\mathrm{Ham}(M,\omega)) \to QH^0(M;\Lambda)^\times$ is a homomorphism.

$S(u) * S(-u) = S(u + (-u)) = 1.$
Therefore, at least one term on the LHS survives.

- McDuff–Tolman developed a way to compute $S(u)$.
  In general $S(u)$ has infinitely many terms.
  $\implies$ It is hard to see which term will survive on the LHS.
- If u is a semifree action whose maximum has codimension 2, some unwanted terms in $S(u)$ vanish.

We can find A, α_i such that $\mathrm{GW}^M_{A,k}([pt], \ldots, \alpha_k) \neq 0$, $\omega(A) = \lambda$.

$\implies w_G \leq \lambda$ by Gromov's theorem.

Remarks

Let (M, ω) be a symplectic toric manifold whose moment polytope is $P \subset \mathbb{R}^n$. Suppose that there exists a primitive vector $u \in \mathbb{Z}^n$ satisfying the following two conditions.

1. $\langle u, \eta \rangle \in \{0, \pm 1\}$ for any primitive η parallel to an edge of P.
2. P has supporting hyperplanes of the form $\{x \in \mathbb{R}^n \mid \langle x, u \rangle \leq \lambda\}$ and $\{x \in \mathbb{R}^n \mid \langle x, u \rangle \geq \mu\}$.

Then the Gromov width of (M, ω) is at most $\lambda - \mu$.

For general nestohedra,

- (1) is true but (2) is not.
- Even when (2) is true, the minimal distance might not be obtained from a singleton, so the formula will be different.