The Gromov width of symplectic toric manifolds associated with graphs

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March 24, 2021
Toric Topology 2021 in Osaka
1. Symplectic toric manifold
2. Nestohedron
3. Computation of Gromov width
Hamiltonian torus action

- \((M^{2n}, \omega)\): symplectic manifold of dimension 2n
- \(T^k \cong (\mathbb{R}/\mathbb{Z})^k\): \(k\)-dimensional torus acting on \(M\)
- \(\mathfrak{t} \cong \mathbb{R}^k\): Lie algebra of \(T^k\) with dual Lie algebra \(\mathfrak{t}^*\)
- \(X\): fundamental vector field on \(M\) for given \(X \in \mathfrak{t}\)

**Definition**

\(T^k\)-action on \((M, \omega)\) is called **Hamiltonian** if for each \(X \in \mathfrak{t}\) there exists \(\mu : M \to \mathfrak{t}^*\) such that

\[
\omega(-, X) = d\langle \mu, X \rangle.
\]

Such \(\mu\) is called a **moment map**.
Symplectic toric manifold

Nestohedron

Computation of Gromov width

Hamiltonian torus action

- \((M^{2n}, \omega)\): symplectic manifold of dimension \(2n\)
- \(T^k \cong (\mathbb{R}/\mathbb{Z})^k\): \(k\)-dimensional torus acting on \(M\)
- \(t \cong \mathbb{R}^k\): Lie algebra of \(T^k\) with dual Lie algebra \(t^*\)
- \(X\): fundamental vector field on \(M\) for given \(X \in t\)

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Fix basis \(X_1, \ldots, X_k\) of \(t^\mathbb{Z}\). This identifies \((t^*, t^*_\mathbb{Z})\) with \((\mathbb{R}^n, \mathbb{Z}^n)\).

Any quantity should be measured with respect to the lattice.
Examples of moment maps

1. $T^2$ acts on $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ by
   $(t_1, t_2) \cdot (z_1, z_2) := (e^{2\pi it_1}z_1, e^{2\pi it_2}z_2).$

   \[
   \mu(z_1, z_2) = (\pi|z_1|^2, \pi|z_2|^2).
   \]

2. $\sigma$: volume form on $S^2$ so that $\int_{S^2} \sigma = 1.$
   $T^2$ acts on $(S^2 \times S^2, a\sigma + b\sigma)$ diagonally.
A symplectic manifold \((M^{2n}, \omega)\) equipped with an effective Hamiltonian \(T^n\)-action is called toric.

In toric case, the image of a moment map \(\mu: M \to \mathbb{R}^n\) is a polytope satisfying the following conditions:

1. There are \(n\) edges at each vertex \(p\).
2. The edges at \(p\) has the form \(p + tu_i\) for some basis vectors \(u_1, \ldots, u_n \in \mathbb{Z}^n\).

Such polytopes are called Delzant polytopes.

**Theorem (Delzant)**

There is a 1-1 correspondence

\[
\{ \text{closed symplectic toric manifolds} \} \leftrightarrow \{ \text{Delzant polytopes} \}.
\]

Want: recover geometric data from Delzant polytopes
**Gromov width**

**Theorem (Gromov nonsqueezing)**

\[ B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2} \text{ symplectically if and only if } r \leq R. \]

(Caution: this picture is far from a proof.)

**Definition (Gromov width)**

\[ w_G(M^{2n}, \omega) := \sup \left\{ \pi r^2 \mid B^{2n}(r) \hookrightarrow (M, \omega) \text{ symplectically} \right\}. \]
These two are different as symplectic toric manifolds, but they are isomorphic as symplectic manifolds.

- Delzant polytope = symplectic structure + torus action
  \[\implies\] symplectic data should be independent of the action.
- In these examples, the Gromov width is the height.
Nestohedron

- A building set $\mathcal{B}$ on $[n + 1]$ is a collection of subsets in $[n + 1]$ satisfying the following conditions:
  1. Each singleton is an element of $\mathcal{B}$.
  2. $I, J \in \mathcal{B}$ with $I \cap J \neq \emptyset \implies I \cup J \in \mathcal{B}$.
- For $I \in \mathcal{B}$, let $\Delta_I := \text{conv}\{e_i | i \in I\}$.
- The nestohedron $P_\mathcal{B}$ is defined to be the Minkowski sum
  \[ P_\mathcal{B} := \sum_{I \in \mathcal{B}} \Delta_I. \]

**Example ($n = 2$, $\mathcal{B} = \{1, 2, 3, 12, 13, 23, 123\}$)**

\[
\begin{align*}
P_\mathcal{B} &= e_1 + e_2 + e_3 \\
&\quad + \text{conv}\{e_1, e_2\} + \text{conv}\{e_1, e_3\} + \text{conv}\{e_2, e_3\} \\
&\quad + \text{conv}\{e_1, e_2, e_3\}
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Properties

- \( P_B \subset \{ \sum_{i=1}^{n+1} x_i = |B| \} \).
- \( P_B \) is a Delzant polytope.
- If \([n + 1] \in B\), then \( P_B \) is \( n \)-dimensional.
  ([\(n + 1\) \(\notin B\), then \( P_B \) is a product of smaller nestohedra.)
- \( I \subset [n + 1] \) defines a facet of \( P_B \):

\[
F_I \subset \left\{ \sum_{i \in I} x_i = |B|_I \right\}, \quad \text{where} \quad B|_I = \{ J \in B \mid J \subset I \}.
\]
Graph associahedron

Building set from a simple graph

- $G$: simple graph with vertex set $[n + 1]$.
- $\mathcal{B}(G) := \{ I \subset [n + 1] \mid \text{the subgraph } G|_I \text{ induced by } I \text{ is connected} \}$.  

$P_{\mathcal{B}(G)}$ is called a graph associahedron.

$G = \begin{array}{c}
1 \\
2 \\
3 
\end{array}$

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(2,1,3) (1,2,3) (1,2,4) (1,4,2) (1,4,1) (2,1,4) (2,3,1) (3,1,2) (3,2,1) (4,1,2) (4,2,1)
Main theorem

Theorem

Let $G$ be a simple graph with the vertex set $[n + 1]$. The Gromov width of the symplectic toric manifold for $P_{B(G)}$ is

$$\min \{k_i > 1 \mid i = 1, \ldots, n + 1\} - 1,$$

where $k_i$ is the number of connected induced subgraphs of $G$ containing $i$.

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$k_1 = k_3 = 3, \quad k_2 = 4.$

$\implies w_G = 2.$
For $I \in \mathcal{B}$, $F_I$ has a parallel facet if and only if $[n+1] \setminus I \in \mathcal{B}$.

When $\mathcal{B}$ is obtained from a graph,

1. Such $I \in \mathcal{B}$ always exists.
2. Minimal distance between such facets are attained when $I$ or its complement is a singleton.
3. $k_i - 1$ is the distance between $F\{i\}$ and $F_{[n+1]\{i\}}$. 
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Distance between parallel facets bounds the Gromov width.
Stabilized embedding

- Gromov nonsqueezing: $w_G(S^2(\pi r^2) \times \mathbb{R}^2) = \pi r^2$.
- On the other hand, $w_G(\Sigma_g(\pi r^2) \times \mathbb{R}^2) = \infty$ for $g \geq 1$.

Let $M_1, M_2$ be symplectic toric manifolds. Is it true that
\[ w_G(M_1 \times M_2) = \min \{ w_G(M_1), w_G(M_2) \}? \]
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Let \( M_1, M_2 \) be symplectic toric manifolds. Is it true that

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\]

For \( M_G \) constructed from graph \( G \), \( w_G(M_G \times \mathbb{R}^2) = w_G(M_G) \).

**Corollary**

*Let \( H \) be a subgraph of \( G \). Suppose \( k = |G| - |H| > 0 \). Then*

\[
M_G \times \mathbb{R}^{2m} \hookrightarrow M_H \times \mathbb{R}^{2k+2m}
\]

*can never be symplectic for any \( m \geq 0 \).*

Is there a topological obstruction to this embedding?
Idea of proof

**Theorem**

Let $G$ be a simple graph with the vertex set $[n + 1]$. The Gromov width of the symplectic toric manifold for $P_{\mathcal{B}(G)}$ is

$$\lambda := \min \{ k_i > 1 \mid i = 1, \ldots, n + 1 \} - 1,$$

where $k_i$ is the number of connected induced subgraphs of $G$ containing $i$.

- (Lower bound) To show $w_G \geq \lambda$,
  1. Use global action-angle coordinates given by moment map.
  2. Find some shape (corresponding to a ball) of “size” $\lambda$ inside $P$.

- (Upper bound) To show $w_G \leq \lambda$,
  1. Find $J$-holomorphic sphere with symplectic area $\leq \lambda$.
  2. Use McDuff–Tolman computation on Seidel representation to find suitable nonvanishing Gromov–Witten invariant.
  3. Use semifree circle action with codimension 2 extrema.
Lower bound

Let $P$ be the moment map image of a symplectic toric manifold.

**Theorem (Mandini–Pabiniak, Latschev–McDuff–Schlenk)**

$$\Diamond^n(\rho) \subset P \implies w_G \geq \rho.$$ 

Find $L_1, \ldots, L_n$ satisfying

- $L_1 \cap \cdots \cap L_n$ is a point.
- Primitive vectors parallel to $L_i$ form a basis for $\mathbb{Z}^n$.
- $L_i$ has affine length $\rho$.

$$\Diamond^n(\rho) := \text{conv}(L_1, \ldots, L_n).$$
Let $P$ be the moment map image of a symplectic toric manifold.

Theorem (Mandini–Pabiniak, Latschev–McDuff–Schlenk)

$$\Diamond^n(\rho) \subset P \implies w_G \geq \rho.$$
Finding ♦ inside $P$

- $G$: connected simple graph with the vertex set $[n + 1]$.
- $B$: building set constructed from $G$.
- $k_i$: number of connected induced subgraphs of $G$ containing $i$.

Assume $k_{n+1}$ is minimal among $k_i$.

Regard $P_B$ as a subset in $\mathbb{R}^n$ by forgetting last coordinate.

**Goal:** find $♦^n(k_{n+1} - 1)$ in $P_B$

Take $L_i = \{(a, \ldots, a, x, a, \ldots, a) \mid 1 \leq x \leq k_{n+1}\}$, where

$$a := \frac{|B| - k_{n+1} - 1}{n - 1}.$$ 

Checking $L_i \subset P_B$ and $1 \leq a \leq k_{n+1}$ reduces to the following.

**Lemma**

$$n \cdot k_i \geq |B| - 1 \quad \text{for any } i = 1, \ldots, n + 1.$$
(Upper bound) To show $w_G \leq \lambda$,

1. Find $J$-holomorphic sphere with symplectic area $\leq \lambda$.
2. Use Seidel representation to find nonzero GW-invariant.
3. Use semifree circle action with codimension 2 extrema.

The (genus zero) Gromov–Witten invariant

$$\text{GW}^M_{A,k}(\alpha_1, \ldots, \alpha_k) \in \mathbb{Q}$$

counts number of $J$-holomorphic spheres in class $A \in H_2(M, \mathbb{Z})$, passing through cycles $\alpha_i \in H^*(M)$.

**Theorem (Gromov)**

If $\text{GW}^M_{A,k}([pt], \alpha_2, \ldots, \alpha_k) \neq 0$ for some $A \in H_2(M, \mathbb{Z})$, $\alpha_i \in H^*(M)$, then the Gromov width of $(M, \omega)$ is at most $\omega(A) > 0$. 

Seidel representation

Goal: find $A, \alpha_i$ such that $GW_{A,k}^M([pt], \ldots, \alpha_k) \neq 0$, $\omega(A) = \lambda$.

- The Seidel morphism is a group homomorphism

$$S : \pi_1(\text{Ham}(M, \omega)) \to (QH^0(M; \Lambda)^\times, \ast).$$

- $QH^\bullet(M; \Lambda) = H^\bullet(M) \otimes \Lambda$ with quantum product $\ast$.
- $\Lambda = \{ \sum a_i q^{\mu_i} t^{\kappa_i} \mid \deg q = 2, \deg t = 0, \text{some condition} \}$.
- $a \ast b = \sum_{A \in H_2(M, \mathbb{Z})} (a \ast b)_A \otimes q^{c_1(A)} t^{\omega(A)}$, where for all $c$,

$$\int_M (a \ast b)_A \cup c = GW^M_{A,3}(a, b, c).$$

We obtain information on GW invariants by studying $S$. 
Upper bound

• \( u \in \pi_1(\text{Ham}(M, \omega)) \) is represented by Hamiltonian \( S^1 \)-action.  
  \[ \implies -u \text{ is represented by the opposite } S^1 \text{-action.} \]

• \( S : \pi_1(\text{Ham}(M, \omega)) \rightarrow QH^0(M; \Lambda)^\times \) is a homomorphism.

\[ S(u) \ast S(-u) = S(u + (-u)) = 1. \]

Therefore, at least one term on the LHS survives.

• McDuff–Tolman developed a way to compute \( S(u) \).
  In general \( S(u) \) has infinitely many terms.
  \[ \implies \text{It is hard to see which term will survive on the LHS.} \]

• If \( u \) is a semifree action whose maximum has codimension 2, some unwanted terms in \( S(u) \) vanish.

We can find \( A, \alpha_i \) such that \( \text{GW}_{A,k}^M([pt], \ldots, \alpha_k) \neq 0, \omega(A) = \lambda. \)

\[ \implies w_G \leq \lambda \text{ by Gromov's theorem.} \]
Remarks

Let \((M, \omega)\) be a symplectic toric manifold whose moment polytope is \(P \subset \mathbb{R}^n\). Suppose that there exists a primitive vector \(u \in \mathbb{Z}^n\) satisfying the following two conditions.

1. \(\langle u, \eta \rangle \in \{0, \pm 1\}\) for any primitive \(\eta\) parallel to an edge of \(P\).
2. \(P\) has supporting hyperplanes of the form \(\{x \in \mathbb{R}^n | \langle x, u \rangle \leq \lambda\}\) and \(\{x \in \mathbb{R}^n | \langle x, u \rangle \geq \mu\}\). Then the Gromov width of \((M, \omega)\) is at most \(\lambda - \mu\).

For general nestohedra,

- (1) is true but (2) is not.
- Even when (2) is true, the minimal distance might not be obtained from a singleton, so the formula will be different.