Strong cohomological rigidity problem of a Hirzebruch surface bundle

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A Bott tower of height $n$ is an iterated $\mathbb{CP}^1$-bundle

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \{\text{a point}\}$$

such that each fibration $B_j \to B_{j-1}$ is a projectivization of rank 2 decomposable vector bundle

$$B_j = P(\xi_j \oplus \xi'_j) \to B_{j-1}.$$

The manifold in the tower is called a Bott manifold.

1. $B_0$ is a point. $B_1 = P(\mathbb{C} \oplus \mathbb{C}) = \mathbb{CP}^1$.

2. $B_2$ is a Hirzebruch surface. Depends on the choice of $\xi_2, \xi'_2$. 
Problem (Cohomological rigidity problem)
Let $B_n, B_n'$ be Bott manifolds of dimension $n$.

\[ H^*(B_n, \mathbb{Z}) \cong H^*(B_n', \mathbb{Z}) \quad ? \quad B_n \cong B'_n \]

Problem (Strong cohomological rigidity problem)
Let $B_n, B_n'$ be Bott manifolds of dimension $n$. For any isomorphism $\varphi: H^*(B_n', \mathbb{Z}) \to H^*(B_n, \mathbb{Z})$,

\[ \exists f: B_n \to B_n' : \text{diffeomorphism s.t. } f^* = \varphi \]

Strong cohomological rigidity $\implies$ Cohomological rigidity.
Strong cohomological rigidity problem can be separated into


2. For any Bott manifold $B_n$ and $\varphi: H^*(B_n, \mathbb{Z}) \to H^*(B_n, \mathbb{Z})$,

   $$\exists f: B_n \to B_n : \text{diffeomorphism s.t. } f^* = \varphi?$$

1 and 2 are true $\implies$ Strong cohomological rigidity.
The followings are useful to show diffeomorphisms.

**Proposition**

For a line bundle $\gamma$, $P(\xi \oplus \xi') \cong P(\gamma \otimes (\xi \oplus \xi'))$ as bundles.

**Proposition**

Let $B$ be a Bott manifold. Rank 2 decomposable vector bundles over $B$ are distinguished by their total Chern classes.

$\gamma$: tautological line bundle over $\mathbb{C}P^1$.

$$B_2 = P(\gamma^{\otimes a} \oplus \gamma^{\otimes b})$$

$$\cong P(\mathbb{C} \oplus \gamma^{\otimes (b-a)})$$

$$\cong \begin{cases} P(\gamma^{\otimes (-k)} \oplus \gamma^{\otimes k}) & \text{if } b-a = 2k, \\ P(\gamma^{\otimes (-k)} \oplus \gamma^{\otimes (k+1)}) & \text{if } b-a = 2k+1, \end{cases}$$

$$\cong \begin{cases} P(\mathbb{C} \oplus \mathbb{C}) & \text{if } b-a \text{ is even} \\ P(\mathbb{C} \oplus \gamma) & \text{if } b-a \text{ is odd.} \end{cases}$$
Introduction

Algebra automorphisms

Equivariant diffeomorphisms of $\Sigma_\alpha$

Remarks

How to attack Strong cohomological rigidity

Proposition

Let $P = P(\mathbb{C} \oplus \xi) \to B$ be a $\mathbb{CP}^1$-bundle. Assume that $H^{\text{odd}}(B) = 0$.

1. $H^*(P)$ is freely generated by $x := c_1(\gamma)$ as an $H^*(B)$-module, where $\gamma$ is the tautological line bundle.

2. $x(-x + c_1(\xi)) = 0$ because $\gamma \oplus \gamma^\perp = \mathbb{C} \oplus \xi$.

3. An automorphism $\varphi : H^*(P) \to H^*(P)$ as an $H^*(B)$-algebra is either $\text{id}$ or $\varphi(x) = -x + c_1(\xi)$. The latter one is induced by $f : P \to P$, $\ell \mapsto \ell^\perp$.

Theorem (2011)

Let $B_n$ and $B'_n$ be Bott manifolds. Let $\varphi : H^*(B'_n) \to H^*(B_n)$ be an isomorphism which is represented by an upper triangular matrix with respect to certain generators. Then $\varphi$ is induced by a diffeomorphism $f : B_n \to B'_n$. 
The purpose of this talk is the following:

**Theorem**

Let $B$ be a Bott manifold. Consider the Hirzebruch surface bundle

$$E = P(\mathbb{C} \oplus \xi_2) \to P(\mathbb{C} \oplus \xi_1) \to B,$$

where $\xi_1$ is a $\mathbb{C}$-line bundle over $B$ and $\xi_2$ is a $\mathbb{C}$-line bundle over $P(\mathbb{C} \oplus \xi_1)$. Let $\tilde{\varphi}: H^*(E) \to H^*(E)$ be an automorphism as an $H^*(B)$-algebra. Then there exists a bundle automorphism $\tilde{f}: E \to E$ over $B$ such that $\tilde{f}^* = \tilde{\varphi}$.
Let $B$ be a Bott manifold. Consider the Hirzebruch surface bundle

$$E = P(\mathbb{C} \oplus \xi_2) \xrightarrow{\pi_2} P(\mathbb{C} \oplus \xi_1) \xrightarrow{\pi_1} B,$$

where $\xi_1$ is a $\mathbb{C}$-line bundle over $B$ and $\xi_2$ is a $\mathbb{C}$-line bundle over $P(\mathbb{C} \oplus \xi_1)$. Let $\tilde{\varphi} : H^*(E) \to H^*(E)$ be an automorphism as an $H^*(B)$-algebras.

- $\pi_1^*, \pi_2^*$ are injective. $H^*(B) \subset H^*(P(\mathbb{C} \oplus \xi_1)) \subset H^*(E)$.
- Let $\gamma_1$ and $\gamma_2$ be the tautological line bundles of $P(\mathbb{C} \oplus \xi_1)$ and $P(\mathbb{C} \oplus \xi_2) = E$, respectively. $x_1 := c_1(\gamma_1)$ and $x_2 := c_1(\gamma_2)$ are generators of $H^*(E)$ as an $H^*(B)$-algebra.
- $c_1(\xi_2) = ax_1 + y$ for some $a \in \mathbb{Z}$ and $y \in H^2(B)$.
- The fiber is a Hirzebruch surface $\Sigma_a = P(\mathbb{C} \oplus \gamma \otimes^a) \to \mathbb{C}P^1$.
- $H^*(E)/H^0(B) \cong H^*(\Sigma_a)$. $\tilde{\varphi}$ descends to $\varphi : H^*(\Sigma_a) \to H^*(\Sigma_a)$. 
\begin{itemize}
  \item $H^*(E)/H^{>0}(B) \cong H^*(\Sigma_a)$. $\widetilde{\varphi} : H^*(E) \to H^*(E)$ descends to an automorphism $\varphi : H^*(\Sigma_a) \to H^*(\Sigma_a)$.

However, not every $\varphi$ can lift. If $\varphi$ lifts an automorphism $\widetilde{\varphi} : H^*(E) \to H^*(E)$, then $c_1(\xi_1)$ and $c_1(\xi_2) = ax_1 + y$ should satisfy a certain condition depending on $\varphi$.

Our plan to attack the problem is

1. Choose $\varphi : H^*(\Sigma_a) \to H^*(\Sigma_a)$.

2. Obtain a necessary and sufficient condition about $c_1(\xi_1)$ and $c_1(\xi_2) = ax_1 + y$.

3. Using the condition, construct $\tilde{f} : E \to E$ such that $\tilde{f}^* = \widetilde{\varphi}$.
Let $\bar{x}_j \in H^2(\Sigma_a)$ be the image of $x_j$ by $H^*(E) \to H^*(\Sigma_a)$. $H^*(\Sigma_a)$ is generated by $\bar{x}_1, \bar{x}_2$ and represented as

$$H^*(\Sigma_a) = \mathbb{Z}[\bar{x}_1, \bar{x}_2]/(\bar{x}_1^2, \bar{x}_2(\bar{x}_2 - a\bar{x}_1)).$$

Primitive square zero elements are

- $\pm \bar{x}_1$ and $\pm(\bar{x}_2 - \frac{a}{2}\bar{x}_1)$ if $a$ is even.
- $\pm \bar{x}_1$ and $\pm(2\bar{x}_2 - a\bar{x}_1)$ if $a$ is odd.

There are 8 automorphisms of $H^*(\Sigma_a)$. 4 of them have upper triangular representation matrices w.r.t. $\bar{x}_1, \bar{x}_2$;

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix},
\begin{pmatrix}
1 & a \\
0 & -1 \\
\end{pmatrix},
\begin{pmatrix}
-1 & a \\
0 & 1 \\
\end{pmatrix}.
$$

If the representation matrix of $\varphi$ is one of above and $\varphi$ lifts $\tilde{\varphi} : H^*(E) \to H^*(E)$, then there exists $\tilde{f} : E \to E$ such that $\tilde{f}^* = \tilde{\varphi}$.
Remaining 4 autohomomorphism have different forms by the parity of $a$.

- In case when $a$ is even, representation matrices are

$$\pm \begin{pmatrix} \frac{a}{2} & \frac{a^2}{4} - 1 \\ -1 & -\frac{a}{2} \end{pmatrix}, \quad \pm \begin{pmatrix} \frac{a}{2} & \frac{a^2}{4} + 1 \\ -1 & -\frac{a}{2} \end{pmatrix}.$$  

- In case when $a$ is odd, representation matrices are

$$\pm \begin{pmatrix} a & \frac{a^2-1}{2} \\ -2 & -a \end{pmatrix}, \quad \pm \begin{pmatrix} a & \frac{a^2+1}{2} \\ -2 & -a \end{pmatrix}.$$
Suppose $a = 0$ and the representation matrix of $\varphi$ is not upper triangular. Then $\varphi$ lifts $\tilde{\varphi}$ if and only if

- $c_1(\xi_1) \pm y$ is even,
- $c_1(\xi_1)^2 = y^2$.

For $x$, $\gamma^x$ denotes a line bundle whose first Chern class is $x$. Since $a = 0$, $E$ is a fiber product

$$
\begin{array}{ccc}
E & \longrightarrow & P(C \oplus \gamma^y) \\
\downarrow & & \downarrow \\
P(C \oplus \xi_1) & \longrightarrow & B
\end{array}
$$

By the condition,

$$
P(C \oplus \gamma^y) \cong P(\sqrt[2]{c_1(\xi_1)-y} \oplus \sqrt[2]{c_1(\xi_1)+y}) \cong P(C \oplus \gamma^{c_1(\xi_1)}).
$$
The automorphism $\tilde{\varphi}$ is induced by one of

$$P(\mathbb{C} \oplus \gamma^c_1(\xi_1))^2 \supset E \ni (\ell_1, \ell_2) \mapsto \begin{cases} (\ell_2, \ell_1), \\ (\ell_2^\perp, \ell_1), \\ (\ell_2, \ell_1^\perp), \\ (\ell_2^\perp, \ell_1^\perp). \end{cases}$$
Suppose that $a$ is nonzero even and the representation matrix of $\varphi$ is not upper triangular. Then $\varphi$ lifts $\widetilde{\varphi}$ if and only if

- $\frac{2\pm a}{4}c_1(\xi_1)$ is integral,
- $y = -\frac{a}{2}c_1(\xi_1),$
- $(4 - a^2)c_1(\xi_1)^2 = 0.$

By the condition,

$$P(\mathbb{C} \oplus \xi_2) \cong P(\mathbb{C} \oplus \gamma^{ax_1 + y})$$

$$\cong P(\gamma^{-\frac{a}{2}x_1} \oplus \gamma^{\frac{a}{2}(x_1 - c_1(\xi_1))})$$

$$\cong P(\mathbb{C} \oplus \gamma^y)$$

as bundles over $P(\mathbb{C} \oplus \xi_1)$. The existence of $\tilde{f}$ follows from the case when $a = 0.$
Suppose that $a$ is odd and the representation matrix of $\varphi$ is not upper triangular. Then $\varphi$ lifts $\bar{\varphi}$ if and only if

- $c_1(\xi_1)$ is even,
- $y = -\frac{a}{2} c_1(\xi_1)$,
- $(1 - a^2)c_1(\xi_1)^2 = 0$.

In case when $a \neq \pm 1$, $c_1(\xi_1)^2 = y^2 = 0$, by an argument similar to the even case we have $E \to B$ is trivial $\Sigma_a$-bundle. Since $\Sigma_a$ is strongly cohomological rigid, $\bar{\varphi}$ is induced by a bundle automorphism $\tilde{f} : E \to E$.

Remaining problem is the case $a = \pm 1$, and we need a different approach.
Suppose that $a = \pm 1$ and the representation matrix of $
abla : H^*(\Sigma_a) \to H^*(\Sigma_a)$ is not upper triangular. Then it is one of

$$\pm \begin{pmatrix} a & 0 \\ -2 & -a \end{pmatrix}, \quad \pm \begin{pmatrix} a & 1 \\ -2 & -a \end{pmatrix}.$$

If $\nabla$ lifts $\widetilde{\nabla} : H^*(E) \to H^*(E)$ then

- $c_1(\xi_1)$ is even,
- $y = -\frac{a}{2} c_1(\xi_1)$,
- $(1-a^2)c_1(\xi_1)^2 = 0$.

$y = -\frac{a}{2} c_1(\xi_1)$ means that the structure group of $E \to B$ can be reduced to $S^1 \acts \Sigma_a$. Thus it is enough to show that

$$\exists f : \Sigma_a \to \Sigma_a \text{ } S^1\text{-equivariant s.t. } f^* = \varphi.$$

($f$ lifts $\tilde{f}$ and $\tilde{f}^* = \varphi$).
\[ \Sigma_a \text{ is diffeomorphic to } S^3 \times S^3 / \sim_a, \text{ where} \]
\[
\left( \left( \begin{array}{c} z_1 \\ w_1 \end{array} \right), \left( \begin{array}{c} z_2 \\ w_2 \end{array} \right) \right) \sim_a \left( \left( \begin{array}{c} z'_1 \\ w'_1 \end{array} \right), \left( \begin{array}{c} z'_2 \\ w'_2 \end{array} \right) \right)
\]

if and only if
\[
\exists s_1, s_2 \in S^1 \text{ s.t. } \left( \left( \begin{array}{c} z_1 \\ w_1 \end{array} \right), \left( \begin{array}{c} z_2 \\ w_2 \end{array} \right) \right) = \left( \left( \begin{array}{c} s_1 z'_1 \\ s_1 w'_1 \end{array} \right), \left( \begin{array}{c} s_2 z'_2 \\ s_1^{-a} s_2 w'_2 \end{array} \right) \right).
\]

The \( S^1 \)-action w.r.t. \( y = -\frac{a}{2} c_1(\xi_1) \) is
\[
t \cdot \left[ \left( \begin{array}{c} z_1 \\ w_1 \end{array} \right), \left( \begin{array}{c} z_2 \\ w_2 \end{array} \right) \right]_{-a} = \left[ \left( \begin{array}{c} z_1 \\ t^2 w_1 \end{array} \right), \left( \begin{array}{c} z_2 \\ t^{-a} w_2 \end{array} \right) \right]_{-a}
\]
For $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, we define $z^{(k)}$ to be

$$z^{(k)} := \begin{cases} 
  z^k & \text{if } k > 0, \\
  1 & \text{if } k = 0, \\
  \overline{z}^{-k} & \text{if } k < 0.
\end{cases}$$

The map $f: \Sigma_a \rightarrow \Sigma_a$ given by

$$\begin{bmatrix} (z_1) \ , \ (z_2) \\ (w_1) \ , \ (w_2) \end{bmatrix}_{-a} \mapsto \left( |z_2|^4 + |w_2|^4 \right)^{-\frac{1}{2}} \begin{bmatrix} z_1 \overline{z}_2^{(-2a)} - \overline{w}_1 w_2^{(-2a)} \\ w_1 z_2^{(-2a)} + \overline{z}_1 \overline{w}_2^{(-2a)} \end{bmatrix}, \begin{bmatrix} \overline{z}_2 \\ w_2 \end{bmatrix}_{-a}$$

is well-defined and a diffeomorphism, because

$$\left( |z_2|^4 + |w_2|^4 \right)^{-\frac{1}{2}} \begin{bmatrix} z_1 \overline{z}_2^{(-2a)} - \overline{w}_1 w_2^{(-2a)} \\ w_1 z_2^{(-2a)} + \overline{z}_1 \overline{w}_2^{(-2a)} \end{bmatrix}$$

is the first column vector of the special unitary matrix

$$\begin{bmatrix} z_1 & -\overline{w}_1 \\ w_1 & \overline{z}_1 \end{bmatrix} \left( |z_2|^4 + |w_2|^4 \right)^{-\frac{1}{2}} \begin{bmatrix} z_2^{(-2a)} & -w_2^{(-2a)} \\ w_2^{(2a)} & z_2^{(2a)} \end{bmatrix}.$$
• By direct computation, \( f \) is \( S^1 \)-equivariant and \( f^*(\overline{x_1}) = -2ax_2 + x_1 \).

• Let \( g_1 : \Sigma_a \to \Sigma_a \) be the equivariant diffeomorphism given by

\[
\begin{pmatrix}
(z_1, w_1) \\
(z_2, w_2)
\end{pmatrix}
\mapsto
\begin{pmatrix}
(z_1, \frac{-w_2}{z_2})
\end{pmatrix}
\]  

The representation matrix of \( g_1^* \) is \( \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix} \).

• Let \( g_2 : \Sigma_a \to \Sigma_a \) be the equivariant diffeomorphism given by

\[
\begin{pmatrix}
(z_1, w_1) \\
(z_2, w_2)
\end{pmatrix}
\mapsto
\begin{pmatrix}
(-\frac{w_1}{z_1}, z_2)
\end{pmatrix}
\]  

The representation matrix of \( g_2^* \) is either \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) or \( \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \).

• \( f, f \circ g_1, f \circ g_2, f \circ g_1 \circ g_2 \) are what we wanted.
One can show the following:

**Theorem**

Let $B$ be a Bott manifold and $E, E' \to B$ Hirzebruch surface bundles. If $H^*(E) \cong H^*(E')$ as $H^*(B)$-algebras, then $E \cong E'$ as bundles over $B$.

Thus we have

**Corollary**

Let $B$ be a Bott manifold and $E, E' \to B$ Hirzebruch surface bundles. Let $\tilde{\varphi}: H^*(E) \to H^*(E')$ be an isomorphism as $H^*(B)$-algebras. Then there exists an isomorphism $\tilde{f}: E' \to E$ as bundles over $B$ such that $\tilde{f}^* = \tilde{\varphi}$. 