Strong cohomological rigidity problem of a Hirzebruch surface bundle

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A Bott tower of height n is an iterated $\mathbb{C}P^1$-bundle

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \{\text{a point}\}$$

such that each fibration $B_j \to B_{j-1}$ is a projectivization of rank 2 decomposable vector bundle

$$B_j = P(\xi_j \oplus \xi_j') \to B_{j-1}.$$

The manifold in the tower is called a Bott manifold.

1. B_0 is a point. $B_1 = P(\mathbb{C} \oplus \mathbb{C}) = \mathbb{C}P^1$.
2. B_2 is a Hirzebruch surface. Depends on the choice of ξ_2, ξ_2'.

Problem (Cohomological rigidity problem)

Let B_n, B'_n be Bott manifolds of dimension n.

$$H^*(B_n, \mathbb{Z}) \cong H^*(B'_n, \mathbb{Z}) \overset{?}{\Longrightarrow} B_n \cong B'_n$$

Problem (Strong cohomological rigidity problem)

Let B_n, B'_n be Bott manifolds of dimension n. For any isomorphism $\varphi\colon H^*(B'_n, \mathbb{Z}) \to H^*(B_n, \mathbb{Z})$,

$$^{\exists} f : B_n \to B'_n : \text{ diffeomorphism s.t. } f^* = \varphi?$$

Strong cohomological rigidity $\Longrightarrow$ Cohomological rigidity.

Strong cohomological rigidity problem can be separated into

1. Cohomological rigidity problem.
2. For any Bott manifold B_n and $\varphi\colon H^*(B_n, \mathbb{Z}) \to H^*(B_n, \mathbb{Z})$,

$$^{\exists} f\colon B_n \to B_n : \text{diffeomorphism s.t. } f^* = \varphi?$$

1 and 2 are true $\Longrightarrow$ Strong cohomological rigidity.

The followings are useful to show diffeomorphisms.

Proposition

For a line bundle γ, $P(\xi \oplus \xi') \cong P(\gamma \otimes (\xi \oplus \xi'))$ as bundles.

Proposition

Let B be a Bott manifold. Rank 2 decomposable vector bundles over B are distinguished by their total Chern classes.

γ : tautological line bundle over $\mathbb{C}P^1$.

$$
\begin{aligned}
B_2 &= P(\gamma^{\otimes a} \oplus \gamma^{\otimes b}) \\
&\cong P(\underline{\mathbb{C}} \oplus \gamma^{\otimes (b-a)}) \\
&\cong \begin{cases} P(\gamma^{\otimes(-k)} \oplus \gamma^{\otimes k}) & \text{if } b-a = 2k, \\ P(\gamma^{\otimes(-k)} \oplus \gamma^{\otimes(k+1)}) & \text{if } b-a = 2k+1, \end{cases} \\
&\cong \begin{cases} P(\underline{\mathbb{C}} \oplus \underline{\mathbb{C}}) & \text{if } b-a \text{ is even} \\ P(\underline{\mathbb{C}} \oplus \gamma) & \text{if } b-a \text{ is odd.} \end{cases}
\end{aligned}
$$

Proposition

Let $P = P(\underline{\mathbb{C}} \oplus \xi) \to B$ be a $\mathbb{C}P^1$-bundle. Assume that $H^{\mathrm{odd}}(B) = 0$.

1. $H^*(P)$ is freely generated by $x := c_1(\gamma)$ as an $H^*(B)$-module, where γ is the tautological line bundle.
2. $x(-x + c_1(\xi)) = 0$ because $\gamma \oplus \gamma^\perp = \underline{\mathbb{C}} \oplus \xi$.
3. An automorphism $\varphi\colon H^*(P) \to H^*(P)$ as an $H^*(B)$-algebra is either id or $\varphi(x) = -x + c_1(\xi)$. The latter one is induced by $f\colon P \to P$, $\ell \mapsto \ell^\perp$.

Theorem (2011)

Let B_n and B_n' be Bott manifolds. Let $\varphi\colon H^*(B_n') \to H^*(B_n)$ be an isomorphism which is represented by an upper triangular matrix with respect to certain generators. Then φ is induced by a diffeomorphism $f\colon B_n \to B_n'$.

The purpose of this talk is the following:

Theorem

Let B be a Bott manifold. Consider the Hirzebruch surface bundle

$$E = P(\underline{\mathbb{C}} \oplus \xi_2) \to P(\underline{\mathbb{C}} \oplus \xi_1) \to B,$$

where ξ_1 is a $\mathbb{C}$-line bundle over B and ξ_2 is a $\mathbb{C}$-line bundle over $P(\underline{\mathbb{C}} \oplus \xi_1)$. Let $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$ be an automorphism as an $H^*(B)$-algebra. Then there exists a bundle automorphism $\widetilde{f}\colon E \to E$ over B such that $\widetilde{f}^* = \widetilde{\varphi}$.

Let B be a Bott manifold. Consider the Hirzebruch surface bundle

$$E = P(\underline{\mathbb{C}} \oplus \xi_2) \xrightarrow{\pi_2} P(\underline{\mathbb{C}} \oplus \xi_1) \xrightarrow{\pi_1} B,$$

where ξ_1 is a $\mathbb{C}$-line bundle over B and ξ_2 is a $\mathbb{C}$-line bundle over $P(\underline{\mathbb{C}} \oplus \xi_1)$. Let $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$ be an automorphism as an $H^*(B)$-algebras.

- π_1^*, π_2^* are injective. $H^*(B) \subset H^*(P(\underline{\mathbb{C}} \oplus \xi_1)) \subset H^*(E)$.
- Let γ_1 and γ_2 be the tautological line bundles of $P(\underline{\mathbb{C}} \oplus \xi_1)$ and $P(\underline{\mathbb{C}} \oplus \xi_2) = E$, respectively. $x_1 := c_1(\gamma_1)$ and $x_2 := c_1(\gamma_2)$ are generators of $H^*(E)$ as an $H^*(B)$-algebra.
- $c_1(\xi_2) = ax_1 + y$ for some $a \in \mathbb{Z}$ and $y \in H^2(B)$.
- The fiber is a Hirzebruch surface $\Sigma_a = P(\underline{\mathbb{C}} \oplus \gamma^{\otimes a}) \to \mathbb{C}P^1$.
- $H^*(E)/H^{>0}(B) \cong H^*(\Sigma_a)$. $\widetilde{\varphi}$ descends to $\varphi\colon H^*(\Sigma_a) \to H^*(\Sigma_a)$.

- $H^*(E)/H^{>0}(B) \cong H^*(\Sigma_a)$. $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$ descends to an automorphism $\varphi\colon H^*(\Sigma_a) \to H^*(\Sigma_a)$.

However, not every φ can lift. If φ lifts an automorphism $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$, then $c_1(\xi_1)$ and $c_1(\xi_2) = ax_1 + y$ should satisfy a certain condition depending on φ.

Our plan to attack the problem is

1. Choose $\varphi\colon H^*(\Sigma_a) \to H^*(\Sigma_a)$.
2. Obtain a necessary and sufficient condition about $c_1(\xi_1)$ and $c_1(\xi_2) = ax_1 + y$.
3. Using the condition, construct $\widetilde{f}\colon E \to E$ such that $\widetilde{f}^* = \widetilde{\varphi}$.

Let $\overline{x_j} \in H^2(\Sigma_a)$ be the image of x_j by $H^*(E) \to H^*(\Sigma_a)$. $H^*(\Sigma_a)$ is generated by $\overline{x_1}, \overline{x_2}$ and represented as

$$H^*(\Sigma_a) = \mathbb{Z}[\overline{x_1}, \overline{x_2}]/(\overline{x_1}^2, \overline{x_2}(\overline{x_2} - a\overline{x_1})).$$

Primitive square zero elements are

- $\pm\overline{x_1}$ and $\pm(\overline{x_2} - \frac{a}{2}\overline{x_1})$ if a is even.
- $\pm\overline{x_1}$ and $\pm(2\overline{x_2} - a\overline{x_1})$ if a is odd.

There are 8 automorphisms of $H^*(\Sigma_a)$. 4 of them have upper triangular representation matrices w.r.t. $\overline{x_1}, \overline{x_2}$;

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}.$$

If the representation matrix of φ is one of above and φ lifts $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$, then there exists $\widetilde{f}\colon E \to E$ such that $\widetilde{f}^* = \widetilde{\varphi}$.

Remaining 4 autohomorphism have different forms by the parity of a.

- In case when a is even, representation matrices are

$$\pm\begin{pmatrix} \frac{a}{2} & \frac{a^2}{4}-1 \\ -1 & -\frac{a}{2} \end{pmatrix}, \quad \pm\begin{pmatrix} \frac{a}{2} & \frac{a^2}{4}+1 \\ -1 & -\frac{a}{2} \end{pmatrix}.$$

- In case when a is odd, representation matrices are

$$\pm\begin{pmatrix} a & \frac{a^2-1}{2} \\ -2 & -a \end{pmatrix}, \quad \pm\begin{pmatrix} a & \frac{a^2+1}{2} \\ -2 & -a \end{pmatrix}.$$

Suppose $a = 0$ and the representation matrix of φ is not upper triangular. Then φ lifts $\widetilde{\varphi}$ if and only if

- $c_1(\xi_1) \pm y$ is even,
- $c_1(\xi_1)^2 = y^2$.

For x, γ^x denotes a line bundle whose first Chern class is x.
Since $a = 0$, E is a fiber product

$$\begin{tikzcd} E \arrow[r] \arrow[d] & P(\underline{\mathbb{C}} \oplus \gamma^y) \arrow[d] \\ P(\underline{\mathbb{C}} \oplus \xi_1) \arrow[r] & B \end{tikzcd}$$

By the condition,

$$P(\underline{\mathbb{C}} \oplus \gamma^y) \cong P(\gamma^{\frac{1}{2}(c_1(\xi_1)-y)} \oplus \gamma^{\frac{1}{2}(c_1(\xi_1)+y)}) \cong P(\underline{\mathbb{C}} \oplus \gamma^{c_1(\xi_1)}).$$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & P(\underline{\mathbb{C}} \oplus \gamma^{c_1(\xi_1)}) \arrow[d] \\ P(\underline{\mathbb{C}} \oplus \gamma^{c_1(\xi_1)}) \arrow[r] & B \end{tikzcd}$$

The automorphism $\widetilde{\varphi}$ is induced by one of

$$P(\underline{\mathbb{C}} \oplus \gamma^{c_1(\xi_1)})^2 \supset E \ni (\ell_1, \ell_2) \mapsto \begin{cases} (\ell_2, \ell_1), \\ (\ell_2^\perp, \ell_1), \\ (\ell_2, \ell_1^\perp), \\ (\ell_2^\perp, \ell_1^\perp). \end{cases}$$

Suppose that a is nonzero even and the representation matrix of φ is not upper triangular. Then φ lifts $\widetilde{\varphi}$ if and only if

- $\frac{2\pm a}{4}c_1(\xi_1)$ is integral,
- $y = -\frac{a}{2}c_1(\xi_1)$,
- $(4-a^2)c_1(\xi_1)^2 = 0$.

By the condition,

$$\begin{aligned} P(\underline{\mathbb{C}} \oplus \xi_2) &\cong P(\underline{\mathbb{C}} \oplus \gamma^{ax_1+y}) \\ &\cong P(\gamma^{-\frac{a}{2}x_1} \oplus \gamma^{\frac{a}{2}(x_1 - c_1(\xi_1))}) \\ &\cong P(\underline{\mathbb{C}} \oplus \gamma^{y}) \end{aligned}$$

as bundles over $P(\underline{\mathbb{C}} \oplus \xi_1)$. The existence of $\widetilde{f}$ follows from the case when $a = 0$.

Suppose that a is odd and the representation matrix of φ is not upper triangular. Then φ lifts $\widetilde{\varphi}$ if and only if

- $c_1(\xi_1)$ is even,
- $y = -\frac{a}{2}c_1(\xi_1)$,
- $(1-a^2)c_1(\xi_1)^2 = 0$.

In case when $a \neq \pm 1$, $c_1(\xi_1)^2 = y^2 = 0$, by an argument similar to the even case we have $E \to B$ is trivial Σ_a-bundle. Since Σ_a is strongly cohomological rigid, $\widetilde{\varphi}$ is induced by a bundle automorphism $\widetilde{f}\colon E \to E$.

Remaining problem is the case $a = \pm 1$, and we need a different approach.

Suppose that $a = \pm 1$ and the representation matrix of $\varphi\colon H^*(\Sigma_a) \to H^*(\Sigma_a)$ is not upper triangular. Then it is one of

$$\pm\begin{pmatrix} a & 0 \\ -2 & -a \end{pmatrix}, \quad \pm\begin{pmatrix} a & 1 \\ -2 & -a \end{pmatrix}.$$

If φ lifts $\widetilde{\varphi}\colon H^*(E) \to H^*(E)$ then

- $c_1(\xi_1)$ is even,
- $y = -\frac{a}{2}c_1(\xi_1)$,
- $\cancel{(1-a^2)c_1(\xi_1)^2 = 0}$.

$y = -\frac{a}{2}c_1(\xi_1)$ means that the structure group of $E \to B$ can be reduced to $S^1 \curvearrowright \Sigma_a$. Thus it is enough to show that

$$^\exists f\colon \Sigma_a \to \Sigma_a \text{ } S^1\text{-equivariant s.t. } f^* = \varphi.$$

(f lifts $\widetilde{f}$ and $\widetilde{f}^* = \widetilde{\varphi}$).

Σ_a is diffeomorphic to $S^3 \times S^3 / \sim_{-a}$, where

$$\left(\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \right) \sim_{-a} \left(\begin{pmatrix} z_1' \\ w_1' \end{pmatrix}, \begin{pmatrix} z_2' \\ w_2' \end{pmatrix} \right)$$

if and only if

$$^{\exists} s_1, s_2 \in S^1 \text{ s.t. } \left(\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \right) = \left(\begin{pmatrix} s_1 z_1' \\ s_1 w_1' \end{pmatrix}, \begin{pmatrix} s_2 z_2' \\ s_1^{-a} s_2 w_2' \end{pmatrix} \right).$$

The S^1-action w.r.t. $y = -\frac{a}{2} c_1(\xi_1)$ is

$$t \cdot \left[\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \right]_{-a} = \left[\begin{pmatrix} z_1 \\ t^2 w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ t^{-a} w_2 \end{pmatrix} \right]_{-a}$$

For $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, we define $z^{(k)}$ to be

$$z^{(k)} := \begin{cases} z^k & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \overline{z}^{-k} & \text{if } k < 0. \end{cases}$$

The map $f\colon \Sigma_a \to \Sigma_a$ given by

$$\left[\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}\right]_{-a} \mapsto \left[(|z_2|^4 + |w_2|^4)^{-\frac{1}{2}} \begin{pmatrix} z_1 z_2^{(-2a)} - \overline{w_1} w_2^{(-2a)} \\ w_1 z_2^{(-2a)} + \overline{z_1} w_2^{(-2a)} \end{pmatrix}, \begin{pmatrix} \overline{z_2} \\ w_2 \end{pmatrix}\right]_{-a}$$

is well-defined and a diffeomorphism, because $(|z_2|^4 + |w_2|^4)^{-\frac{1}{2}} \begin{pmatrix} z_1 z_2^{(-2a)} - \overline{w_1} w_2^{(-2a)} \\ w_1 z_2^{(-2a)} + \overline{z_1} w_2^{(-2a)} \end{pmatrix}$ is the first column vector of the special unitary matrix

$$\begin{pmatrix} z_1 & -\overline{w_1} \\ w_1 & \overline{z_1} \end{pmatrix} (|z_2|^4 + |w_2|^4)^{-\frac{1}{2}} \begin{pmatrix} z_2^{(-2a)} & -w_2{}^{(-2a)} \\ w_2^{(2a)} & z_2{}^{(2a)} \end{pmatrix}.$$

- By direct computation, f is S^1-equivariant and $f^*(\overline{x_1}) = -2a\overline{x_2} + \overline{x_1}$.
- Let $g_1\colon \Sigma_a \to \Sigma_a$ be the equivariant diffeomorphism given by

$$\left[\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}\right]_{-a} \mapsto \left[\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} -\overline{w_2} \\ \overline{z_2} \end{pmatrix}\right]_{-a}.$$

  The representation matrix of g_1^* is $\begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$.
- Let $g_2\colon \Sigma_a \to \Sigma_a$ be the equivariant diffeomorphism given by

$$\left[\begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}\right]_{-a} \mapsto \left[\begin{pmatrix} -\overline{w_1} \\ \overline{z_1} \end{pmatrix}, \begin{pmatrix} w_2 \\ z_2 \end{pmatrix}\right]_{-a}.$$

  The representation matrix of g_2^* is either $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}$.
- f, $f \circ g_1$, $f \circ g_2$, $f \circ g_1 \circ g_2$ are what we wanted.

One can show the following:

Theorem

Let B be a Bott manifold and $E, E' \to B$ Hirzebruch surface bundles. If $H^*(E) \cong H^*(E')$ as $H^*(B)$-algebras, then $E \cong E'$ as bundles over B.

Thus we have

Corollary

Let B be a Bott manifold and $E, E' \to B$ Hirzebruch surface bundles. Let $\widetilde{\varphi}\colon H^*(E) \to H^*(E')$ be an isomorphism as $H^*(B)$-algebras. Then there exists an isomorphism $\widetilde{f}\colon E' \to E$ as bundles over B such that $\widetilde{f}^* = \widetilde{\varphi}$.