Circle actions on almost complex manifolds with isolated fixed points

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During this talk, we study circle actions on almost complex manifolds that only have isolated fixed points, and discuss classifications when the number of fixed points is small.

Almost complex manifolds

An **almost complex manifold** (M, J) is a manifold M with a bundle map $J : TM \longrightarrow TM$ (called an **almost complex structure**) that restricts to a linear complex structure on each tangent space, i.e., J restricts to a map $J_m : T_m M \longrightarrow T_m M$ such that $J_m^2 = -\mathrm{Id}_{T_m M}$ for each $m \in M$.

A circle action $S^1 \times M \to M$ on an almost complex manifold (M, J) is said to **preserve the almost complex structure**, if $dg \circ J = J \circ dg$ for every $g \in S^1$.

During this talk, any circle action on an almost complex manifold is assumed to preserve the almost complex structure.

If a torus T^k action on a compact almost complex manifold has fixed points, we can find a subcircle S of T^k that has the same fixed point set. In this sense, we can reduce torus actions to circle actions.

Any symplectic circle action on a symplectic manifold (M, ω) preserves an almost complex structure compatible with ω. By definition, any complex manifold is almost complex. Therefore, results for circle actions on almost complex manifolds apply to circle actions on complex manifolds and symplectic manifolds.

Weights at an isolated fixed point

Let the circle S^1 act on a $2n$-dimensional almost complex manifold M. Let p be an isolated fixed point. We can identify T_pM with $\mathbb{C}^n$, on which the circle acts by

$$g \cdot (z_1, \cdots, z_n) = (g^{w_{p,1}} z_1, \cdots, g^{w_{p,n}} z_n)$$

for all $g \in S^1 \subset \mathbb{C}$, $z_i \in \mathbb{C}$, where $w_{p,i}$ are non-zero integers, $1 \leq i \leq n$.

The $w_{p,i}$ are called **weights** (also called **rotation numbers**) at p.

The fixed point data, the collection $\bigcup_{p \in M^{S^1}} \{w_{p,1}, \cdots, w_{p,n}\}$ of the multisets of the weights at the fixed points, encodes information of a manifold, such as the Chern numbers, the Hirzebruch χ_y-genus (and hence the Todd genus, the signature, and the Euler number), etc.

Properties of weights

Let the circle act on a compact almost complex manifold M with isolated fixed points. For each time there is a weight w at some fixed point p, there exists a fixed point q with weight $-w$. For an integer w, let $N_p(w)$ be the number of times the weight w occurs at p, i.e., $N_p(w) = |\{i \mid w_{p,i} = w\}|$.

Lemma (Hattori '85, Li '12)

Let w be an integer. Then

$$\sum_{p \in M^{S^1}} N_p(w) = \sum_{p \in M^{S^1}} N_p(-w).$$

(Hattori used equivariant K-theory and Li used the Atiyah-Singer index formula.)

If a fixed point p has weight w and a fixed point q has weight $-w$, then we can draw an edge e from p to q, assigning a label w to the edge e. In this way, we can associate a (labeled, directed) multigraph to M.

An immediate consequence of the lemma is the following:

Corollary

Let the circle act on a $2n$-dimensional compact almost complex manifold with isolated fixed points. If n is odd, then the number of fixed points cannot be odd.

A **labeled (directed) multigraph** is a set V of vertices, a set E of edges, maps $i\colon E \to V$ and $t\colon E \to V$ giving the initial and terminal vertices of each edge, and a map w from E to the positive integers.

Definition

Let M be a compact almost complex manifold equipped with a circle action having isolated fixed points. We say that a labeled multigraph Γ **describes** M if the following hold:

1. The vertex set of Γ is M^{S^1}.
2. The multiset of weights at p is $\{w(e) \mid i(e) = p\} \cup \{-w(e) \mid t(e) = p\}$ for all $p \in M^{S^1}$; and
3. For each edge e, the two endpoints $i(e)$ and $t(e)$ are in the same component of the isotropy submanifold $M^{\mathbb{Z}/(w(e))}$ (the $\mathbb{Z}/(w(e))$-fixed point set of M; $\mathbb{Z}/(w(e))$ acts on M as a subgroup of S^1).

Therefore, if there is an edge e from p to q, p has weight $w(e)$ and q has weight $-w(e)$. Γ encodes some geometry (isotropy submanifolds) of M.

Proposition

Let the circle act on a compact almost complex manifold M with isolated fixed points. Then there exists a multigraph describing M that does not have any self-loops.

Example: an action on $\mathbb{CP}^2$.

Consider a linear circle action on $\mathbb{CP}^2$ by

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : g^a z_1 : g^{a+b} z_2]$$

for some positive integers a and b. The action has three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$.

Consider $[0 : 1 : 0]$. Since $z_1 \neq 0$, $(\frac{z_0}{z_1}, \frac{z_2}{z_1})$ are local coordinates. Near $[0 : 1 : 0]$, the S^1-action is given by

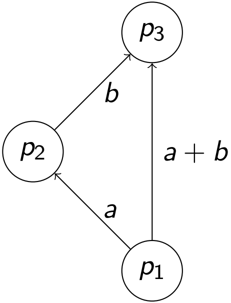
$$g \cdot (\frac{z_0}{z_1}, \frac{z_2}{z_1}) = (\frac{z_0}{g^a z_1}, \frac{g^{a+b} z_2}{g^a z_1}) = (g^{-a} \frac{z_0}{z_1}, g^b \frac{z_2}{z_1}).$$

Hence the weights at $[0 : 1 : 0]$ are $\{-a, b\}$.

The weights at the fixed points are $\{a + b, a\}$, $\{-a, b\}$, and $\{-b, -a - b\}$, respectively.

Multigraph for $\mathbb{CP}^2$

Action: $g \cdot [z_0 : z_1 : z_2] = [z_0 : g^a z_1 : g^{a+b} z_2]$
Fixed points: $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$.
Weights at the fixed points: $\{a + b, a\}$, $\{-a, b\}$, $\{-b, -a - b\}$



p_1 and p_3 are connected by an edge with label $a + b$, which means from definition that p_1 and p_3 are in the same component $[z_0 : 0 : z_2]$ of $M^{\mathbb{Z}_{a+b}}$ that are fixed by $\mathbb{Z}_{a+b}$-action;
$g \cdot [z_0 : 0 : z_2] = [z_0 : 0 : g^{a+b} z_2]$.

The Hirzebruch surface

Fix an integer n. The Hirzebruch surface is a complex surface that is a set of points $([z_0 : z_1 : z_2], [w_1 : w_2])$ in $\mathbb{CP}^2 \times \mathbb{CP}^1$ that satisfies $z_1 w_2^n = z_2 w_1^n$.

For $g \in S^1 \subset \mathbb{C}$, let g act by $g \cdot ([z_0 : z_1 : z_2], [w_1 : w_2]) = ([g^a z_0 : z_1 : g^{nb} z_2], [w_1 : g^b w_2])$.

The action has 4 fixed points, $p_1 = ([1 : 0 : 0], [1 : 0]), p_2 = ([1 : 0 : 0], [0 : 1]), p_3 = ([0 : 1 : 0], [1 : 0]), p_4 = ([0 : 0 : 1], [0 : 1])$.

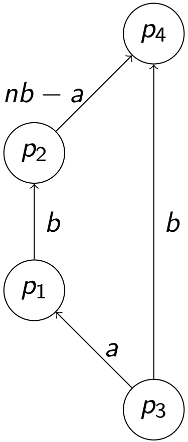
The weights at the fixed points are $\{-a, b\}$, $\{nb - a, -b\}$, $\{a, b\}$, and $\{a - nb, -b\}$ for some positive integers a and b.

Multigraphs for a Hirzebruch surface

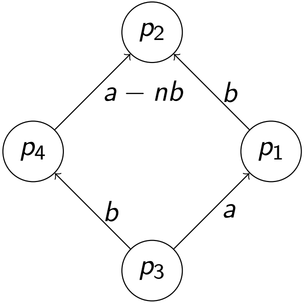
Action:

$g \cdot ([z_0 : z_1 : z_2], [w_1 : w_2]) = ([g^a z_0 : z_1 : g^{nb} z_2], [w_1 : g^b w_2])$

Weights: $\{-a, b\}$, $\{nb - a, -b\}$, $\{a, b\}$, and $\{a - nb, -b\}$



(a) Case 1: $nb - a > 0$



(b) Case 2: $nb - a < 0$

Theorem (J '17)

Let the circle act on a compact almost complex manifold M.

1. *If there is exactly 1 fixed point p, then M is a point itself, that is, $M = \{p\}$.*
2. *If there are exactly 2 fixed points, then either M is the 2-sphere, or $\dim M = 6$ and the weights at the fixed points are $\{-a-b, a, b\}$ and $\{-a, -b, a+b\}$ for some positive integers a and b.*
3. *If there are exactly 3 fixed points, then $\dim M = 4$ and the weights at the fixed points are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some positive integers a and b.*

If there are 2 fixed points, Kosniowski obtained an analogous result for complex manifolds ('74) and Pelayo and Tolman ('11) obtained an analogous result for symplectic manifolds. The case of 3 fixed points is obtained after a careful treatment of a result of J ('14) for symplectic manifolds.

Related conjecture

Kosniowski conjectured that for a circle action on a compact unitary manifold, there is a relationship between the dimension of the manifold and the number of fixed points.

Conjecture (Kosniowski '79)

If the circle acts on a compact unitary manifold M with k fixed points and M does not bound a unitary manifold equivariantly, then $\dim M < 4k$.

An almost complex version of Kosniowski's conjecture can be stated as follows.

Conjecture (Almost complex version)

Let the circle on a compact almost complex manifold M with k fixed points. Then $\dim M < 4k$.

[J] confirms almost complex version of the conjecture for $k \leq 3$.

From now on, we discuss classifications if there are 4 fixed points. First, if $\dim M = 2$, M is a disjoint union of two S^2's.

Theorem (J '20a)

Let the circle act on a 4-dimensional compact almost complex manifold with 4 fixed points. Then the multisets of the weights at the fixed points are $\{a, b\}$, $\{-a, b\}$, $\{-b, nb - a\}$, *and* $\{-b, a - nb\}$ *for some positive integers* a *and* b *and for some integer* n.

In this case, the weights at the fixed points agree with those of the circle actions on the Hizerbruch surfaces described earlier.

Next, we discuss classification of a circle action on a 6-dimensional compact almost complex manifold M with 4 fixed points.

If $\mathrm{Todd}(M) = 1$ and $c_1^3(M)[M] \neq 0$, then Ahara ('91) classified the fixed point data.

On the other hand, If M is symplectic and the action is Hamiltonian, then Tolman ('10) classified the fixed point data. In addition, Tolman determined the cohomology ring and the Chern classes. In this case, the Todd genus is also equal to 1.

I classified the fixed point data without these assumptions. In particular, the Todd genus of M need not be 1.

Theorem (J '20a)

Let the circle act on a 6-dimensional compact almost complex manifold with 4 fixed points. Then exactly one of the following holds for the multisets of the weights at the fixed points.

(1) $\{a,b,c\}$, $\{-a,b-a,c-a\}$, $\{-b,a-b,c-b\}$, $\{-c,a-c,b-c\}$ $(\mathrm{Todd}(M)=1)$

(2) $\{a,a+b,a+2b\}$, $\{-a,b,a+2b\}$, $\{-a-2b,-b,a\}$, $\{-a-2b,-a-b,-a\}$ $(\mathrm{Todd}(M)=1)$

(3) $\{1,2,3\}$, $\{-1,1,a\}$, $\{-1,-a,1\}$, $\{-1,-2,-3\}$ $(\mathrm{Todd}(M)=1)$

(4) $\{-a-b,a,b\}$, $\{-c-d,c,d\}$, $\{-a,-b,a+b\}$, $\{-c,-d,c+d\}$ $(\mathrm{Todd}(M)=0)$

(5) $\{-3a-b,a,b\}$, $\{-2a-b,3a+b,3a+2b\}$, $\{-a,-a-b,2a+b\}$, $\{-b,-3a-2b,a+b\}$ $(\mathrm{Todd}(M)=0)$

(6) $\{-a-b,2a+b,b\}$, $\{-2a-b,a,b\}$, $\{-b,-2a-b,a+b\}$, $\{-a,-b,2a+b\}$ $(\mathrm{Todd}(M)=0)$

From now on, in each case of the theorem we discuss the existence of a manifold.

Type 1: $\{a, b, c\}$, $\{-a, b-a, c-a\}$, $\{-b, a-b, c-b\}$, $\{-c, a-c, b-c\}$

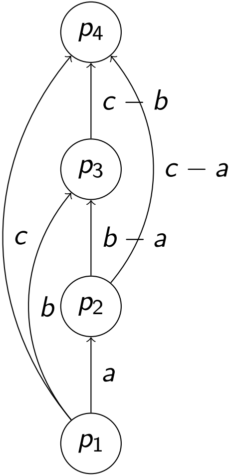
Type 1 is a standard linear action on $\mathbb{CP}^3$;

$$g \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g^a z_1 : g^b z_2 : g^c z_3]$$

for mutually distinct positive integers a, b, and c.

The fixed points are $[1:0:0:0], [0:1:0:0], [0:0:1:0]$, and $[0:0:0:1]$.

Type 1: $\{a, b, c\}$, $\{-a, b-a, c-a\}$, $\{-b, a-b, c-b\}$, $\{-c, a-c, b-c\}$



(a) $\mathbb{CP}^3$ type

Type 2: $\{a, a+b, a+2b\}$, $\{-a, b, a+2b\}$, $\{-a-2b, -b, a\}$, $\{-a-2b, -a-b, -a\}$

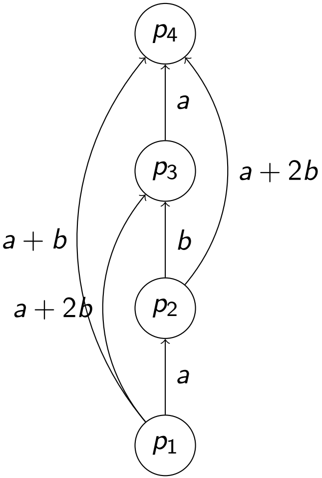
Type 2 is the complex quadric in $\mathbb{CP}^4$.

Let $g \in S^1 \subset \mathbb{C}$ act on the complex quadric $Q^3 = \{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 | z_0 z_2 + z_2 z_3 + z_4^3 = 0\}$ by

$$g \cdot [z_0 : z_1 : z_2 : z_3 : z_4] = [g^a z_0 : g^{-a} z_1 : g^b z_2 : g^{-b} z_3 : z_4]$$

for mutually distinct positive integers a, b. The fixed points are $[1:0:0:0:0], [0:1:0:0:0], [0:0:1:0:0], [0:0:0:1:0]$ and the weights at the fixed points are as above.

Type 2: $\{a, a+b, a+2b\}$, $\{-a, b, a+2b\}$, $\{-a-2b, -b, a\}$, $\{-a-2b, -a-b, -a\}$



(a) Complex quadric

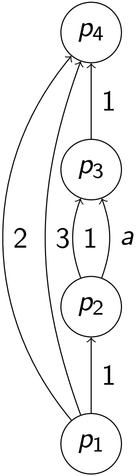
Type 3: $\{1,2,3\}$, $\{-1,1,a\}$, $\{-1,-a,1\}$, $\{-1,-2,-3\}$

If $a=2$ or $a=3$, then Type 3 is included in Type 1 or Type 2, respectively.

If $a=4$ or 5, then Ahara and McDuff respectively constructed such a manifold. (Ahara constructed a complex manifold and McDuff constructed a symplectic manifold.)

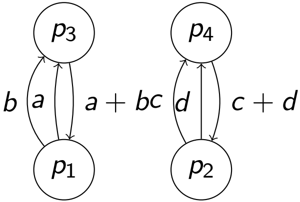
For other values of a, the existence is unknown, to my best knowledge.

Type 3: $\{1, 2, 3\}$, $\{-1, 1, a\}$, $\{-1, -a, 1\}$, $\{-1, -2, -3\}$



(a) Fano 3-fold type

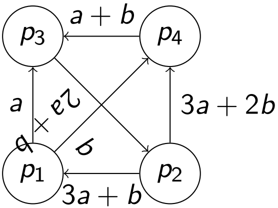
Regarded as $G(2)/SU(3)$, S^6 admits a circle action with 2 fixed points. The weights at the fixed points are $\{-a-b, a, b\}$ and $\{-a, -b, a+b\}$ for some positive integers a and b.
Type 4: $\{-a-b, a, b\}$, $\{-c-d, c, d\}$, $\{-a, -b, a+b\}$, $\{-c, -d, c+d\}$
Type 4 is a disjoint union of rotations on two S^6's.



(a) $S^6 \cup S^6$ type

Type 5: $\{-3a-b, a, b\}$, $\{-2a-b, 3a+b, 3a+2b\}$,
$\{-a, -a-b, 2a+b\}$, $\{-b, -3a-2b, a+b\}$

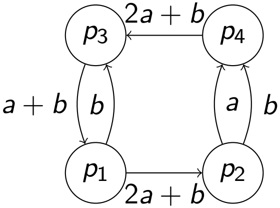
Type 5 is an equivariant blow up of a fixed point, of a rotation of S^6.



(a) Blow up S^6

Type 6: $\{-a-b, 2a+b, b\}$, $\{-2a-b, a, b\}$, $\{-b, -2a-b, a+b\}$, $\{-a, -b, 2a+b\}$

To the author's knowledge, the existence of a manifold of Type 6 is unknown; however, this can be realized as an equivariant blow up of S^2 in a rotation of S^6.



(a) Blow up S^2 in S^6?

If dim $M = 8$ and there are 4 fixed points, all the Chern numbers of M and the Hirzebruch χ_y-genus of M agree with those of $S^2 \times S^6$.

Theorem (J '20-b)

Let the circle act on an 8-dimensional compact almost complex manifold M with 4 fixed points. Then the Hirzebruch χ_y-genus of M is $\chi_y(M) = -y + 2y^2 - y^3$ and the Chern numbers of M are

$$\int_M c_1^4 = \int_M c_1^2 c_2 = \int_M c_2^2 = 0, \text{ and } \int_M c_1 c_3 = \int_M c_4 = 4.$$

In particular, M is unitary cobordant to $S^2 \times S^6$.

Recently, Goertsches, Konstantis, and Zoller announced that there exists an 8-dimensional compact almost complex manifold admitting a circle action with 4 fixed points, which is not $S^2 \times S^6$ (but it is an S^2-bundle over S^6.)

Idea of proof

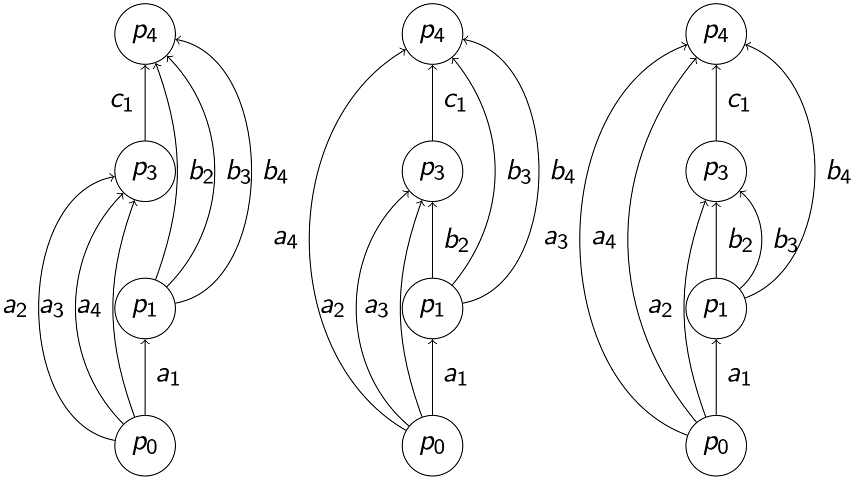
1 fixed point or 2 fixed points: We can use the ABBV localization formula or the Atiyah-Singer index formula for the Dolbeault-type operators on almost complex S^1-manifolds.
ex) If $\dim M > 0$ and $|M^{S^1}| = 1$, then

$$0 = \int_M 1 = \frac{1}{\prod_{i=1}^n w_{p,i}} \neq 0,$$

implying $\dim M = 0$.

3 fixed points: Consider possible multigraphs describing M (This is implicit in my original proof). Next, we consider restrictions on the weights that they must satisfy (the ABBV localization formula or the index formula, etc.) If $\dim M > 4$, then the weights at the fixed points cannot satisfy all of the restrictions, implying $\dim M = 4$.

4 fixed points is similar.



These are some of possible multigraphs if $\dim M = 8$, there are 4 fixed points, and $\mathrm{Todd}(M) = 1$. I show any of these cannot occur as a multigraph describing M, proving $\mathrm{Todd}(M) = 0$.

A diagonal action on $S^6 \times S^6$ provides a 12-dimensional example with 4 fixed points. On the other hand, if $\dim M > 8$ and $\dim M \neq 12$, there are no known examples with 4 fixed points. Therefore, we can ask the following question, inspired by Kosniowski's conjecture.

Question

Does there exist a compact almost complex manifold M equipped with a circle action having 4 fixed points such that $\dim M > 8$ *and* $\dim M \neq 12$*?*

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This is the end of my talk. Thank you very much!