Tverberg’s theorem for cell complexes

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Points in a plane

Theorem (Birch ’59)

Any $3N$ points in a plane determine $N$ triangles which have a point in common.
Points in a plane

Theorem (Birch ’59)

Any $3N$ points in a plane determine $N$ triangles which have a point in common.

Is ”$3N$” tight? — consider convex hulls instead of triangles.
Suppose you have 4 points in a place.

Then you can partition them into 2 subsets whose convex hulls have a point in common.
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Why don’t we replace triangles with convex hulls?
Theorem (Birch ’59)

Any $3N - 2$ points in a plane can be partitioned into $N$ subsets whose convex hulls have a point in common.

"$3N - 2$" is best possible. For example, $3N - 3$ for $N = 2$ does not work.
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Tverberg’s theorem

Theorem (Tverberg ’66)

Any \((d + 1)(r - 1) + 1\) points in \(\mathbb{R}^d\) can be partitioned into \(r\) subsets whose convex hulls have a point in common.
Tverberg’s theorem

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Corollary (Radon ’21)

Any \(d + 2\) points in \(\mathbb{R}^d\) can be partitioned into 2 subsets whose convex hulls have a point in common.
Restatement

\((d + 1)(r - 1) + 1\) points in \(\mathbb{R}^d\) determines an affine map

\[\Delta^{(d+1)(r-1)} \rightarrow \mathbb{R}^d\]

such that convex hulls of points are unions of images of faces.

Moreover, a common point of convex hulls lie in a simplex in each convex hull.

Theorem (Tverberg’s theorem, restated)

*For any affine map \(f : \Delta^{(d+1)(r-1)} \rightarrow \mathbb{R}^d\), there are pairwise disjoint faces \(\sigma_1, \ldots, \sigma_r\) of \(\Delta^{(d+1)(r-1)}\) such that*

\[f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.\]
Topological Tverberg theorem

What happens if an affine map is replaced with a continuous map?
Topological Tverberg theorem

What happens if an affine map is replaced with a continuous map?

Theorem (Bárány, Shlosman, Szűcs ’81, Özaydin ’87, Volovikov ’96)

For any continuous map $f : \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(d+1)(r-1)}$ such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$$

whenever $r$ is a prime power.

Remark

1. The condition that $r$ is a prime power is necessary (Frick ’15).
2. The case $r = 2$ is called the topological Radon theorem.
Question

Why are we still considering a simplex?

- Tverberg asked whether we can replace a simplex by a polytope.
  The answer is YES but the replacement is not essential because the boundary of a polytope is a refinement of the boundary of a simplex.

- Blagojević, Haase and Ziegler ’19 constructed a family of matroids \(\{M_r\}_{r \geq 2}\) which are replaceable with a simplex.

We want more!!
\( r \)-complementary \( n \)-acyclic complex

- For faces \( \sigma_1, \ldots, \sigma_k \) of a regular CW complex \( X \), let
  \[
  X(\sigma_1, \ldots, \sigma_k)
  \]
  be a subcomplex of \( X \) consisting of faces separated from \( \sigma_1, \ldots, \sigma_k \).

- For \( n \geq 0 \), \( X \) is called \( n \)-acyclic if \( \tilde{H}_\ast(X) = 0 \) for \( \ast \leq n \).

- A \((-1)\)-acyclic space will mean a non-empty space.

**Definition** A regular CW complex \( X \) is \( r \)-complementary \( n \)-acyclic if for any faces \( \sigma_1, \ldots, \sigma_k \) with
  \[
  \dim \sigma_1 + \cdots + \dim \sigma_k \leq n + 1 \quad \text{and} \quad 0 \leq k \leq r,
  \]
  \( X(\sigma_1, \ldots, \sigma_k) \) is \((n - \dim \sigma_1 - \cdots - \dim \sigma_k)\)-acyclic.
Examples

Example
A $d$-simplex is $(r - 1)$-complementary $(d - r)$-acyclic.

Proposition

*Every simplicial $d$-sphere is $1$-complementary $(d - 1)$-acyclic.*

Example

Here is a 1-complementary 1-acyclic non-polyhedral 2-sphere.

\[
\begin{tikzpicture}
\end{tikzpicture}
\]
Main theorem

Theorem

Let $X$ be an $(r - 1)$-complementary $(d(r - 1) - 1)$-acyclic regular CW complex where $r$ is a prime power. Then for any continuous map

$$f : X \to \mathbb{R}^d$$

there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $X$ such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$
Generalized the topological Radon theorem

Corollary

Let $X$ be a simplicial $d$-sphere. Then for any continuous map

$$f : X \to \mathbb{R}^d$$

there are disjoint faces $\sigma_1, \sigma_2$ of $X$ such that

$$f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.$$ 

Remark

Since not every simplicial sphere is the boundary of a polytope, this is a proper generalization of the topological Radon theorem.
Discretized configuration space

- Let $X$ be a regular CW complex.

The discretized configuration space

$$\text{Conf}_r(X)$$

is a subspace of $X^r$ consisting of faces $\sigma_1 \times \cdots \times \sigma_r$ such that $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint, where $\sigma_1, \ldots, \sigma_r$ are faces of $X$.

- Let $\Delta = \{(x_1, \ldots, x_r) \in (\mathbb{R}^d)^r \mid x_1 = \cdots = x_r\}$.

**Lemma**

*Let $f : X \to \mathbb{R}^d$ be a continuous map such that for every pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $X$,*

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.$$ 

*Then there is a $\Sigma_r$-equivariant map*

$$\text{Conf}_r(X) \to (\mathbb{R}^d)^r - \Delta.$$
Lemma

If Conf\(_r(X)\) is \((d(r - 1) - 1)\)-acyclic, then for any continuous map

\[
f : X \to \mathbb{R}^d
\]

there are pairwise disjoint faces \(\sigma_1, \ldots, \sigma_r\) of \(X\) such that

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

Proof.

Note that \((\mathbb{R}^d)^r - \Delta \simeq S^{d(r-1)-1}\).

The case \(r\) is a prime.

The actions of \(\mathbb{Z}/r (\subset \Sigma_r)\) on Conf\(_r(X)\) and \((\mathbb{R}^d)^r - \Delta\) are free, so we can apply the Borsuk-Ulam theorem.

The case \(r\) is a prime power.

We need a little bit of computation of equivariant cohomology. \(\square\)
Acyclicity of Conf$_r$(X)

Proposition

If $X$ is $(r-1)$-complementary $n$-acyclic, then Conf$_r$(X) is $n$-acyclic.

Proof.

Step 1 We describe Conf$_r$(X) as a homotopy colimit of a functor over the face poset of $X$.

Step 2 We construct a spectral sequence (≅ Bousfield-Kan spectral sequence) which computes the homology of a homotopy colimit.

Step 3 By induction on $r$, we show that if $X$ is $(r-1)$-complementary $n$-acyclic, then

$$H_\ast(\text{Conf}_r(X)) \cong H_\ast(X) \quad (\ast \leq n)$$

implying Conf$_r$(X) is $n$-acyclic.

The main theorem is obtained by the above lemma and proposition.
Tverberg complex

Definition
A regular CW complex \( X \) is \((d, r)\)-Tverberg if for any continuous map \( f : X \to \mathbb{R}^d \), there are pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( X \) such that

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

Example
If a regular CW complex \( X \) includes a \((d, r)\)-Tverberg subcomplex, then \( X \) itself is \((d, r)\)-Tverberg.

What is an "essential" \((d, r)\)-Tverberg complex?
Atomicity

Definition
A \((d, r)\)-Tverberg complex is called atomic if it has no \((d, r)\)-Tverberg subcomplex and is not a proper refinement of a \((d, r)\)-Tverberg complex.

Problem
Count atomic \((d, r)\)-Tverberg complexes for small \(d, r\).

Proposition
Atomic \((1, 2)\)-Tverberg complexes are a triangle and a Y-shaped graph.

Proposition
The only atomic \((2, 2)\)-Tverberg polyhedral sphere is a tetrahedron.