Tverberg's theorem for cell complexes

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joint work with S. Hasui, M. Takeda and M. Tsutaya

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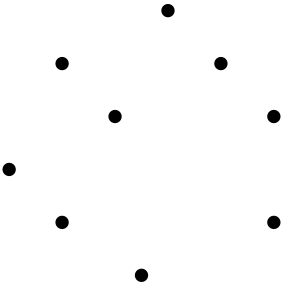
Points in a plane

Theorem (Birch '59)

Any $3N$ points in a plane determine N triangles which have a point in common.



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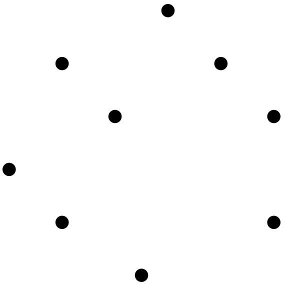
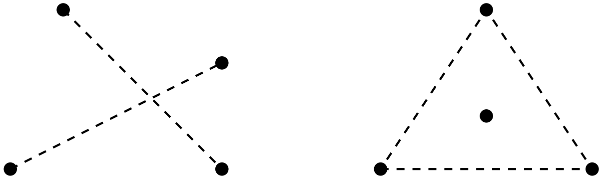


Is "$3N$" tight? — consider convex hulls instead of triangles.

Suppose you have 4 points in a place.

Then you can partition them into 2 subsets whose convex hulls have a point in common.



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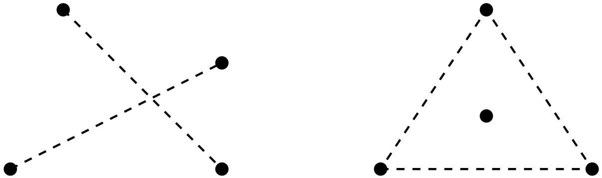

Why don't we replace triangles with convex hulls?

Theorem (Birch '59)

Any $3N - 2$ *points in a plane can be partitioned into* N *subsets whose convex hulls have a point in common.*

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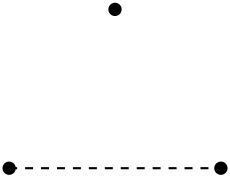
Any $3N - 2$ points in a plane can be partitioned into N subsets whose convex hulls have a point in common.

"$3N - 2$" is best possible. For example, $3N - 3$ for $N = 2$ does not work:



Tverberg's theorem

Theorem (Tverberg '66)

Any $(d+1)(r-1)+1$ *points in* $\mathbb{R}^d$ *can be partitioned into* r *subsets whose convex hulls have a point in common.*

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Corollary (Radon '21)

Any $d+2$ *points in* $\mathbb{R}^d$ *can be partitioned into 2 subsets whose convex hulls have a point in common.*

Restatement

$(d+1)(r-1)+1$ points in $\mathbb{R}^d$ determines an affine map

$$\Delta^{(d+1)(r-1)} \to \mathbb{R}^d$$

such that convex hulls of points are unions of images of faces.

Moreover, a common point of convex hulls lie in a simplex in each convex hull.

Theorem (Tverberg's theorem, restated)

For any affine map $f\colon \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(d+1)(r-1)}$ such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

Topological Tverberg theorem

What happens if an affine map is replaced with a continuous map?

Topological Tverberg theorem

What happens if an affine map is replaced with a continuous map?

Theorem (Bárány, Shlosman, Szűcs '81, Özaydin '87, Volovikov '96)

For any continuous map $f\colon \Delta^{(d+1)(r-1)} \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(d+1)(r-1)}$ such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$$

whenever r is a prime power.

Remark

1. The condition that r is a prime power is necessary (Frick '15).
2. The case $r = 2$ is called the topological Radon theorem.

Question

Why are we still considering a simplex?

- Tverberg asked whether we can replace a simplex by a polytope.

  The answer is YES but the replacement is not essential because the boundary of a polytope is a refinement of the boundary of a simplex.

- Blagojević, Haase and Ziegler '19 constructed a family of matroids $\{M_r\}_{r\geq 2}$ which are replaceable with a simplex.

We want more!!

r-complementary n-acyclic complex

- For faces $\sigma_1, \ldots, \sigma_k$ of a regular CW complex X, let

$$X(\sigma_1, \ldots, \sigma_k)$$

  be a subcomplex of X consisting of faces separated from $\sigma_1, \ldots, \sigma_k$.
- For $n \geq 0$, X is called n-acyclic if $\widetilde{H}_*(X) = 0$ for $* \leq n$.
- A (-1)-acyclic space will mean a non-empty space.

Definition A regular CW complex X is r-complementary n-acyclic if for any faces $\sigma_1, \ldots, \sigma_k$ with

$$\dim \sigma_1 + \cdots + \dim \sigma_k \leq n + 1 \quad \text{and} \quad 0 \leq k \leq r,$$

$X(\sigma_1, \ldots, \sigma_k)$ is $(n - \dim \sigma_1 - \cdots - \dim \sigma_k)$-acyclic.

Examples

Example

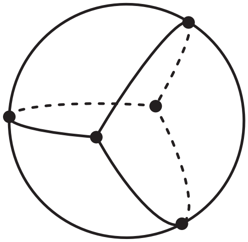
A d-simplex is $(r-1)$-complementary $(d-r)$-acyclic.

Proposition

Every simplicial d-sphere is 1-complementary $(d-1)$-acyclic.

Example

Here is a 1-complementary 1-acyclic non-polyhedral 2-sphere.



Main theorem

Theorem

Let X be an $(r-1)$-complementary $(d(r-1)-1)$-acyclic regular CW complex where r is a prime power. Then for any continuous map

$$f\colon X \to \mathbb{R}^d$$

there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of X such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

Generalized the topological Radon theorem

Corollary

Let X be a simplicial d-sphere. Then for any continuous map

$$f: X \to \mathbb{R}^d$$

there are disjoint faces σ_1, σ_2 of X such that

$$f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.$$

Remark

Since not every simplicial sphere is the boundary of a polytope, this is a proper generalization of the topological Radon theorem.

Discretized configuration space

- Let X be a regular CW complex.

The discretized configuration space

$$\mathrm{Conf}_r(X)$$

is a subspace of X^r consisting of faces $\sigma_1 \times \cdots \times \sigma_r$ such that $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint, where $\sigma_1, \ldots, \sigma_r$ are faces of X.

- Let $\Delta = \{(x_1, \ldots, x_r) \in (\mathbb{R}^d)^r \mid x_1 = \cdots = x_r\}$.

Lemma

Let $f\colon X \to \mathbb{R}^d$ be a continuous map such that for every pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of X,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.$$

Then there is a Σ_r-equivariant map

$$\mathrm{Conf}_r(X) \to (\mathbb{R}^d)^r - \Delta.$$

Lemma

If $\mathrm{Conf}_r(X)$ *is* $(d(r-1)-1)$*-acyclic, then for any continuous map*

$$f : X \to \mathbb{R}^d$$

there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ *of* X *such that*

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

Proof.

Note that $(\mathbb{R}^d)^r - \Delta \simeq S^{d(r-1)-1}$.

The case r is a prime.

The actions of $\mathbb{Z}/r$ $(\subset \Sigma_r)$ on $\mathrm{Conf}_r(X)$ and $(\mathbb{R}^d)^r - \Delta$ are free, so we can apply the Borsuk-Ulam theorem.

The case r is a prime power.

We need a little bit of computation of equivariant cohomology. $\square$

Acyclicity of $\mathrm{Conf}_r(X)$

Proposition

If X is $(r-1)$-complementary n-acyclic, then $\mathrm{Conf}_r(X)$ is n-acyclic.

Proof.

Step 1 We describe $\mathrm{Conf}_r(X)$ as a homotopy colimit of a functor over the face poset of X.

Step 2 We construct a spectral sequence ($\fallingdotseq$ Bousfield-Kan spectral sequence) which computes the homology of a homotopy colimit.

Step 3 By induction on r, we show that if X is $(r-1)$-complementary n-acyclic, then

$$H_*(\mathrm{Conf}_r(X)) \cong H_*(X) \quad (* \leq n)$$

implying $\mathrm{Conf}_r(X)$ is n-acyclic. $\square$

The main theorem is obtained by the above lemma and proposition.

Tverberg complex

Definition

A regular CW complex X is (d, r)-Tverberg if for any continuous map $f : X \to \mathbb{R}^d$, there are pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of X such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

Example

If a regular CW complex X includes a (d, r)-Tverberg subcomplex, then X itself is (d, r)-Tverberg.

What is an "essential" (d, r)-Tverberg complex?

Atomicity

Definition

A (d, r)-Tverberg complex is called atomic if it has no (d, r)-Tverberg subcomplex and is not a proper refinement of a (d, r)-Tverberg complex.

Problem

Count atomic (d, r)-Tverberg complexes for small d, r.

Proposition

Atomic $(1, 2)$-Tverberg complexes are a triangle and a Y-shaped graph.

Proposition

The only atomic $(2, 2)$-Tverberg polyhedral sphere is a tetrahedron.