On string polytopes II

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What are string polytopes?

- $G$: simply-connected semisimple algebraic group over $\mathbb{C}$. Today, $G = SL_{n+1}(\mathbb{C})$.
- $i$: reduced decomposition of the longest element of the Weyl group of $G$.
- $\lambda$: dominant integral weight.

Using these data, one can define the string polytope $\Delta_i(\lambda)$, which

1. lives in $\mathbb{R}^N$, where $N = \dim_{\mathbb{C}} G/B = \frac{n(n+1)}{2}$,
2. $\Delta_i(\lambda) \cap \mathbb{Z}^N \leftrightarrow$ weights of $V(\lambda)$,
3. is a Newton–Okounkov body of $(G/B, \mathcal{L}_\lambda, \nu_i)$ (by [Kaveh, 15]).
4. For $i = (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1)$,

$$\Delta_i(\lambda) \cong \text{Gelfand–Cetlin polytope } GC(\lambda).$$

Combinatorics of $\Delta_i(\lambda)$ depends on $i$. 
Gelfand–Cetlin polytopes

\[ G = \text{SL}_3(\mathbb{C}), \lambda = 2\varpi_1 + 2\varpi_2. \]
String polytopes and symplectic data of the corresponding Lagrangian submanifold

Theorem [Nishinou–Nohara–Ueda, 10]

One can get symplectic topological information (so called disk potential) of $\Phi^{-1}(u)$ using the combinatorics of $\Delta_i(\lambda)$. 
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Gelfand–Cetlin type string polytopes

Small resolutions of string polytopes

Future works

Gelfand–Cetlin toric varieties

$X_{\tilde{\Sigma}}$ is a small desingularization of $X_{\Sigma}$ if $\tilde{\Sigma}$ is smooth and it is a refinement of $\Sigma$ satisfying $\tilde{\Sigma}(1) = \Sigma(1)$.

Note: not all string polytopes are Gelfand–Cetlin polytopes.
There are combinatorially different string polytopes

\[ G = \text{SL}_4(\mathbb{C}) \]

- Classifying the unimodular equivalence classes of string polytopes.
- Finding small Fano toric desingularization.

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String polytopes and wiring diagrams

$G = \text{SL}_{n+1}(\mathbb{C})$.

$\Delta_i(\lambda) = C_i \cap C_i^\lambda$

$C_i$ is called the string cone, $C_i^\lambda$ is called the $\lambda$-cone.
Gleizer–Postnikov’s rigorous paths

\[ i = (2, 1, 3, 2, 3, 1) \]

\[ G(i, 3) \]

- \((\ell_1 \rightarrow \ell_2) \rightsquigarrow C_5\)
- \((\ell_2 \rightarrow \ell_3) \rightsquigarrow C_1, C_2, C_3, C_4.\)
- \((\ell_2 \rightarrow \ell_4 \rightarrow \ell_1 \rightarrow \ell_3) \rightsquigarrow C_2, C_3, C_4.\)
- \((\ell_2 \rightarrow \ell_1 \rightarrow \ell_3) \rightsquigarrow C_2, C_4.\)
- \((\ell_2 \rightarrow \ell_4 \rightarrow \ell_3) \rightsquigarrow C_3, C_4.\)
- \((\ell_2 \rightarrow \ell_1 \rightarrow \ell_4 \rightarrow \ell_3) \rightsquigarrow C_4.\)
- \((\ell_3 \rightarrow \ell_4) \rightsquigarrow C_6.\)
String inequalities

**Definition**

The string inequality associated to $P$ is defined by

$$\sum_{C_j \subset \text{region enclosed by } P} m_j \geq 0$$

\[
\begin{align*}
(l_1 \rightarrow l_2) &\rightsquigarrow C_5 & m_5 \geq 0 \\
(l_2 \rightarrow l_3) &\rightsquigarrow C_1, C_2, C_3, C_4 & m_1 + m_2 + m_3 + m_4 \geq 0 \\
(l_2 \rightarrow l_4 \rightarrow l_1 \rightarrow l_3) &\rightsquigarrow C_2, C_3, C_4 & m_2 + m_3 + m_4 \geq 0 \\
(l_2 \rightarrow l_1 \rightarrow l_3) &\rightsquigarrow C_2, C_4 & m_2 + m_4 \geq 0 \\
(l_2 \rightarrow l_4 \rightarrow l_3) &\rightsquigarrow C_3, C_4 & m_3 + m_4 \geq 0 \\
(l_2 \rightarrow l_1 \rightarrow l_4 \rightarrow l_3) &\rightsquigarrow C_4 & m_4 \geq 0 \\
(l_3 \rightarrow l_4) &\rightsquigarrow C_6 & m_6 \geq 0
\end{align*}
\]

defines $C_i$
Let $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n$ be a dominant weight. The $\lambda$-inequality associated to $m_j$ is defined by

$$\sum_{k \geq j, i_k = i_j} m_k \leq \lambda_{ij}.$$
### Indices

\[ \text{ind}_A(i) = \# \text{ of crossings below } \ell_1, \]
\[ \text{ind}_D(i) = \# \text{ of crossings below } \ell_{n+1}. \]

\[
\begin{array}{c|c|c}
\ell_1 & \ell_2 & \ell_3 & \ell_4 \\
\hline
i = (1, 2, 1, 3, 2, 1) & i = (2, 1, 3, 2, 3, 1) \\
\hline
\text{ind}_D(i) = 0 & \text{ind}_D(i) = 1 \\
\text{ind}_A(i) = 3 & \text{ind}_A(i) = 1 \\
\end{array}
\]
Contractions

- $C_D(i)$: erase $\ell_{n+1}$ and rearrange.
- $C_A(i)$: erase $\ell_1$ and rearrange.

Contraction maps a reduced word of the longest element in $\mathcal{S}_{n+1}$ to a reduced word of the longest element in $\mathcal{S}_n$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$C_A(i)$</th>
<th>$C_D(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_1 \ell_2 \ell_3 \ell_4$</td>
<td>$\ell_1 \ell_2 \ell_3$</td>
<td>$\ell_1 \ell_2 \ell_3$</td>
</tr>
</tbody>
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Gelfand–Cetlin type string polytopes

Theorem [Cho–Kim–Lee–Park 1, 19+]

Let $i$ be a reduced word of the longest element in $S_{n+1}$. Let $\lambda$ be a regular dominant integral weight. Then the following are equivalent.

1. The string polytope $\Delta_i(\lambda)$ is unimodularly equivalent to the Gelfand–Cetlin polytope $GC(\lambda)$.

2. The string polytope $\Delta_i(\lambda)$ has exactly $n(n + 1)$ many facets.

3. The associated string cone $C_i$ is simplicial.

4. There exists a sequence $(\sigma_1, \ldots, \sigma_n) \in \{A, D\}^n$ such that

$$\text{ind}_{\sigma_k} (C_{\sigma_{k+1}} \circ \cdots \circ C_{\sigma_n}(i)) = 0$$

for all $k = n, \ldots, 1$.

Here $\text{ind}$ denotes the $\bullet$-index of $i$ and $C_{\bullet}$ denotes a $\bullet$-contraction where $\bullet = D$ or $A$. 
Examples

- $i = (2, 1, 2, 3, 2, 1)$. Then
  
  \[
  \text{ind}_D(i) = 0, \quad C_D(i) = (2, 1, 2), \\
  \text{ind}_A(2, 1, 2) = 0, \quad C_A(2, 1, 2) = (1), \\
  \text{ind}_D(1) = 0.
  \]

  Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \simeq \text{GC}(\lambda)$.

- $i = (2, 1, 3, 2, 3, 1)$. Then
  
  \[
  \text{ind}_A(2, 1, 3, 2, 3, 1) = 1, \quad \text{ind}_D(2, 1, 3, 2, 3, 1) = 1.
  \]

  Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \not\simeq \text{GC}(\lambda)$. Indeed,

  \[
  \# \text{ of facets of } \Delta_i(\lambda) = 13 \neq 12.
  \]
As one may see, the combinatorics of string polytopes heavily depend on that of wiring diagrams. By analyzing wiring diagrams, [Cho–Kim–L, 20+] enumerates the number $g_{c(n)}$ of Gelfand–Cetlin type reduced words for $G = \text{SL}_{n+1}(\mathbb{C})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{c(n)}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>40</td>
<td>916</td>
<td>102176</td>
<td>68464624</td>
</tr>
<tr>
<td># of all reduced decomp.</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>16</td>
<td>768</td>
<td>292864</td>
<td>1100742656</td>
<td>48608795688960</td>
</tr>
</tbody>
</table>

Table: The first few terms of $g_{c(n)}$ (cf. OEIS A337699).
Reduction decompositions having small indices

Definition

A reduced decomposition $i$ of the longest element in $\mathfrak{S}_{n+1}$ has small indices if there exists a sequence $(\sigma_1, \ldots, \sigma_n) \in \{A, D\}^n$ such that

$$\text{ind}_{\sigma_n}(i) \leq \kappa(\sigma_{n-1}, \sigma_n),$$

$$\text{ind}_{\sigma_k}(C_{\sigma_{k+1}} \circ \cdots \circ C_{\sigma_n}(i)) = 0 \quad \text{for all} \quad k = n - 1, \ldots, 1,$$

where $\kappa(\sigma_{n-1}, \sigma_n)$ is 2 if $\sigma_{n-1} = \sigma_n; n - 1$ otherwise.

For example, $i = (1, 3, 2, 1, 3, 2)$ has small indices. Take $(D, D, D) \in \{A, D\}^3$. Then,

$$\text{ind}_D(1, 3, 2, 1, 3, 2) = 2,$$

$$C_D(i) = (1, 2, 1, \_) \rightsquigarrow \text{ind}_D(1, 2, 1) = 0,$$

$$C_D(1, 2, 1) = (1) \rightsquigarrow \text{ind}_D(1) = 0.$$

All reduced decompositions of the longest element in $\mathfrak{S}_4$ have small indices.
Theorem [Cho–Kim–Lee–Park 2, 19⁺]

Let $i$ be a reduced decomposition of the longest element in $S_{n+1}$. Let $\lambda$ be a regular dominant integral weight. If $i$ has small indices, then $X_{\Delta_i}(\lambda)$ admits a small toric desingularization $\tilde{X}$. Moreover, $\tilde{X}$ is obtained by blowing-ups of a Bott manifold.

Corollary

Suppose that $i$ has small indices. Then, the following holds.

1. $\Delta_i(\lambda)$ is integral for any dominant integral weight $\lambda$.
2. For a parabolic subgroup $P$, $\Delta_i(\lambda_P)$ is reflexive, where $\lambda_P$ is the weight corresponding to the anticanonical line bundle of $G/P$.
3. One can compute the Floer theoretical disk potential defined by Fukaya–Oh–Ohta–Ono of the Lagrangian submanifold in $G/B$ given by $\Delta_i(\lambda)$ for any regular dominant integral weight $\lambda$. 
1. Studying which topological/geometric data can be obtained from different $\Delta_i(\lambda)$.

- The combinatorial relations among string polytopes (Berenstein–Zelevinsky, 01) and other Newton–Okounkov bodies of $G/B$ (Fujita–Higashitani, 20$^+$) have been studied.

There is an open embedding $U_{w_0}^- \hookrightarrow G/B$ and the the unipotent cell $U_{w_0}^-$ admits a cluster algebra structure. [Fujita–Oya, 20$^+$] constructed $\Delta(G/B, L_\lambda, \nu_s)$ for each seed $s$ and proved that $\Delta(G/B, L_\lambda, \nu_s) \simeq \Delta_i(\lambda)$ when $s$ comes from $i$. 
$G = \text{SL}_4(\mathbb{C})$

(1, 2, 3, 1, 2, 1)  
(1, 3, 2, 3, 1, 2)  
(1, 3, 2, 1, 3, 2)  
(3, 1, 2, 1, 3, 2)  
(3, 2, 1, 2, 3, 2)  

(1, 2, 1, 3, 2, 1)  
(2, 1, 3, 2, 3, 1)  
(2, 3, 1, 2, 3, 1)  
(2, 3, 1, 2, 1, 3)  
(2, 3, 2, 1, 2, 3)  

(1, 2, 1, 3, 2, 1)  
(2, 1, 3, 2, 3, 1)  
(2, 3, 1, 2, 3, 1)  
(2, 3, 1, 2, 1, 3)  
(2, 3, 2, 1, 2, 3)  

Question (work in progress)
Describe $\Delta(\mathcal{G}/B, L_\lambda, \nu_s)$ for various seeds $s$ explicitly.
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**Question (work in progress)**

Describe $\Delta(G/B, \mathcal{L}_\lambda, \nu_s)$ for various seeds $s$ explicitly.
Future works

2. Constructing a completely integrable system associated to $\Delta_i(\lambda)$.
   - For the Gelfand–Cetlin polytope, [Guillemin–Sternberg, 83] provided a completely integrable system. Using this, a detailed description of topology of Gelfand–Cetlin fibers has been studied by [Cho–Kim–Oh, 20].
   - [Harada–Kaveh, 15] proved the existence of completely integrable system. However, we don’t know the explicit description yet.

3. Generalizing the previous result to other Lie types.
   - String polytopes (and also Newton–Okounkov bodies) are defined for any semisimple Lie groups. We studied the combinatorics of string polytope only for Lie type $A$. 
Thank you for your attention!