On string polytopes II

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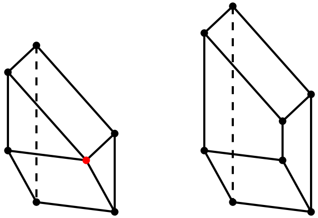

arXiv:1904.00130, arXiv:1912.00658, arXiv:2009.06906

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Toric Topology 2021 in Osaka, Japan

What are string polytopes?

- G: simply-connected semisimple algebraic group over $\mathbb{C}$. Today, $G = \mathsf{SL}_{n+1}(\mathbb{C})$.
- $\mathbf{i}$: reduced decomposition of the longest element of the Weyl group of G.
- λ: dominant integral weight.

Using these data, one can define the string polytope $\Delta_{\mathbf{i}}(\lambda)$, which

1. lives in $\mathbb{R}^N$, where $N = \dim_{\mathbb{C}} G/B = \frac{n(n+1)}{2}$,
2. $\Delta_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^N \leftrightarrow$ weights of $V(\lambda)$,
3. is a Newton–Okounkov body of $(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{i}})$ (by [Kaveh, 15]).
4. For $\mathbf{i} = (1, 2, 1, 3, 2, 1, ..., n, n-1, \dots, 1)$,

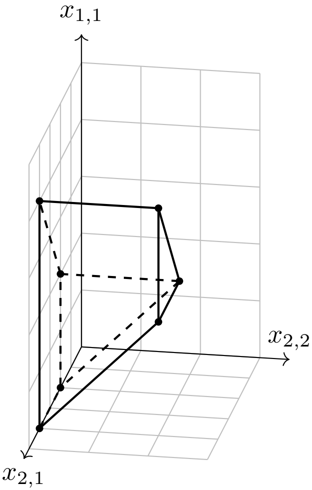
$$\Delta_{\mathbf{i}}(\lambda) \simeq \text{Gelfand–Cetlin polytope } \mathsf{GC}(\lambda).$$

Combinatorics of $\Delta_{\mathbf{i}}(\lambda)$ depends on $\mathbf{i}$.

Gelfand–Cetlin polytopes

$$G = \mathsf{SL}_3(\mathbb{C}), \lambda = 2\varpi_1 + 2\varpi_2.$$

$$\begin{array}{ccccc} \color{red}{4} & & \color{red}{2} & & \color{red}{0} \\ & x_{2,1} & & x_{2,2} & \\ & & x_{1,1} & & \end{array}$$



String polytopes and symplectic data of the corresponding Lagrangian submanifold

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d] \\
\Phi^{-1}(u) \subset & G/B \arrow[r, squiggly, dash, "\text{toric}", "\text{degeneration}"'] \arrow[dr, "\Phi"] & X_{\Delta_{\mathbf{i}}(\lambda)} \arrow[d] \\
 & & \Delta_{\mathbf{i}}(\lambda) \ni u \text{ (interior)}
\end{tikzcd}$$

If $\exists$ small Fano toric desingularization

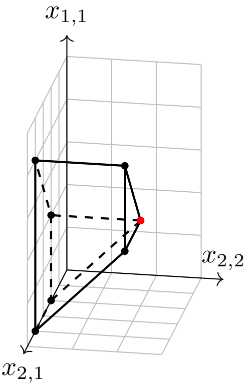
[Harada–Kaveh, 15]

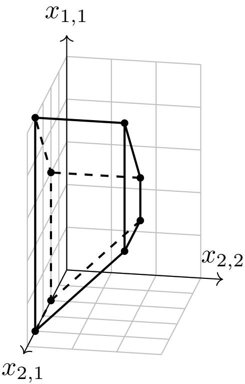
Theorem [Nishinou–Nohara–Ueda, 10]

One can get symplectic topological information (so called disk potential) of $\Phi^{-1}(u)$ using the combinatorics of $\Delta_{\mathbf{i}}(\lambda)$.

Gelfand–Cetlin toric varieties

$X_{\widetilde{\Sigma}}$ is a small desingularization of X_{Σ} if $\widetilde{\Sigma}$ is smooth and it is a refinement of Σ satisfying $\widetilde{\Sigma}(1) = \Sigma(1)$.

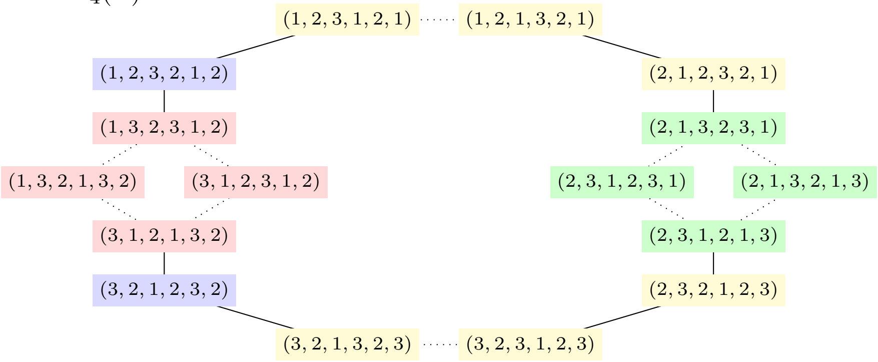




Note: not all string polytopes are Gelfand–Cetlin polytopes.

There are combinatorially different string polytopes

$G = \mathrm{SL}_4(\mathbb{C})$



GOAL

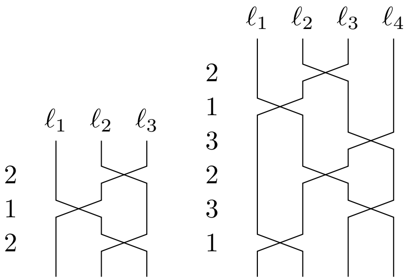
1. Classifying the unimodular equivalence classes of string polytopes.
2. Finding small Fano toric desingularization.

String polytopes and wiring diagrams

$G = \mathrm{SL}_{n+1}(\mathbb{C})$.

$$\Delta_{\mathbf{i}}(\lambda) = C_{\mathbf{i}} \cap C_{\mathbf{i}}^{\lambda}$$

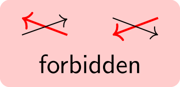
$C_{\mathbf{i}}$ is called the string cone, $C_{\mathbf{i}}^{\lambda}$ is called the λ-cone.

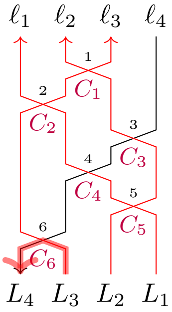


$\mathbf{i} = (2, 1, 2)$ $\mathbf{i} = (2, 1, 3, 2, 3, 1)$

Gleizer–Postnikov's rigorous paths

$\mathbf{i} = (2, 1, 3, 2, 3, 1)$





$G(\mathbf{i}, 3)$

- $(\ell_1 \to \ell_2) \rightsquigarrow C_5$
- $(\ell_2 \to \ell_3) \rightsquigarrow C_1, C_2, C_3, C_4$.
- $(\ell_2 \to \ell_4 \to \ell_1 \to \ell_3) \rightsquigarrow C_2, C_3, C_4$.
- $(\ell_2 \to \ell_1 \to \ell_3) \rightsquigarrow C_2, C_4$.
- $(\ell_2 \to \ell_4 \to \ell_3) \rightsquigarrow C_3, C_4$.
- $(\ell_2 \to \ell_1 \to \ell_4 \to \ell_3) \rightsquigarrow C_4$.
- $(\ell_3 \to \ell_4) \rightsquigarrow C_6$.

String inequalities

Definition

The string inequality associated to P is defined by

$$\sum_{C_j \subset \text{ region enclosed by } P} m_j \geq 0$$

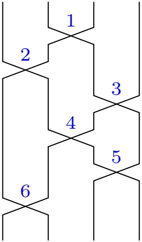
$$\left.\begin{array}{ll}
(\ell_1 \to \ell_2) \rightsquigarrow C_5 & m_5 \geq 0 \\
(\ell_2 \to \ell_3) \rightsquigarrow C_1, C_2, C_3, C_4 & m_1 + m_2 + m_3 + m_4 \geq 0 \\
(\ell_2 \to \ell_4 \to \ell_1 \to \ell_3) \rightsquigarrow C_2, C_3, C_4 & m_2 + m_3 + m_4 \geq 0 \\
(\ell_2 \to \ell_1 \to \ell_3) \rightsquigarrow C_2, C_4 & m_2 + m_4 \geq 0 \\
(\ell_2 \to \ell_4 \to \ell_3) \rightsquigarrow C_3, C_4 & m_3 + m_4 \geq 0 \\
(\ell_2 \to \ell_1 \to \ell_4 \to \ell_3) \rightsquigarrow C_4 & m_4 \geq 0 \\
(\ell_3 \to \ell_4) \rightsquigarrow C_6 & m_6 \geq 0
\end{array}\right\} \text{defines } \mathcal{C}_{\mathbf{i}}$$

λ-inequalities

Definition

Let $\lambda = \lambda_1\varpi_1 + \cdots + \lambda_n\varpi_n$ be a dominant weight. The λ-inequality associated to m_j is defined by

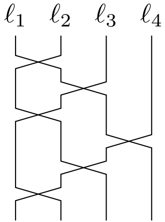
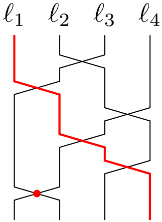
$$\sum_{k \geq j, i_k = i_j} m_k \leq \lambda_{i_j}.$$



$$\left.\begin{array}{l} m_1 + m_4 \leq \lambda_2 \\ m_2 + m_6 \leq \lambda_1 \\ m_3 + m_5 \leq \lambda_3 \\ m_4 \leq \lambda_2 \\ m_5 \leq \lambda_3 \\ m_6 \leq \lambda_1 \end{array}\right\} \text{defines } C_{\mathbf{i}}^{\lambda}$$

Indices

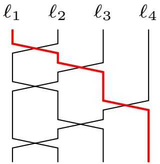
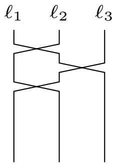
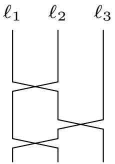
$$\mathrm{ind}_A(\mathbf{i}) = \#\text{ of crossings below } \ell_1,$$
$$\mathrm{ind}_D(\mathbf{i}) = \#\text{ of crossings below } \ell_{n+1}.$$

$\mathbf{i} = (1,2,1,3,2,1)$	$\mathbf{i} = (2,1,3,2,3,1)$
	
$\mathrm{ind}_D(\mathbf{i}) = 0$ $\mathrm{ind}_A(\mathbf{i}) = 3$	$\mathrm{ind}_D(\mathbf{i}) = 1$ $\mathrm{ind}_A(\mathbf{i}) = 1$

Contractions

- $C_D(\mathbf{i})$: erase ℓ_{n+1} and rearrange.
- $C_A(\mathbf{i})$: erase ℓ_1 and rearrange.

> Contraction maps a reduced word of the longest element in $\mathfrak{S}_{n+1}$ to a reduced word of the longest element in $\mathfrak{S}_n$.

$\mathbf{i}$	$C_A(\mathbf{i})$	$C_D(\mathbf{i})$
		

Gelfand–Cetlin type string polytopes

Theorem [Cho–Kim–L–Park 1, 19+]

Let $\mathbf{i}$ be a reduced word of the longest element in $\mathfrak{S}_{n+1}$. Let λ be a regular dominant integral weight. Then the following are equivalent.

1. The string polytope $\Delta_{\mathbf{i}}(\lambda)$ is unimodularly equivalent to the Gelfand–Cetlin polytope GC(λ).
2. The string polytope $\Delta_{\mathbf{i}}(\lambda)$ has exactly $n(n+1)$ many facets.
3. The associated string cone $C_{\mathbf{i}}$ is simplicial.
4. There exists a sequence $(\sigma_1, \dots, \sigma_n) \in \{A, D\}^n$ such that

$$\mathrm{ind}_{\sigma_k}\left(C_{\sigma_{k+1}} \circ \cdots \circ C_{\sigma_n}(\mathbf{i})\right) = 0 \quad \text{for all } k = n, \dots, 1.$$

Here $\mathrm{ind}_\bullet$ denotes the $\bullet$-index of $\mathbf{i}$ and $C_\bullet$ denotes a $\bullet$-contraction where $\bullet = D$ or A.

Examples

- $\mathbf{i} = (2,1,2,3,2,1)$. Then

$$\begin{aligned} \mathrm{ind}_D(\mathbf{i}) &= 0, \quad C_D(\mathbf{i}) = (2,1,2), \\ \mathrm{ind}_A(2,1,2) &= 0, \quad C_A(2,1,2) = (1), \\ \mathrm{ind}_D(1) &= 0. \end{aligned}$$

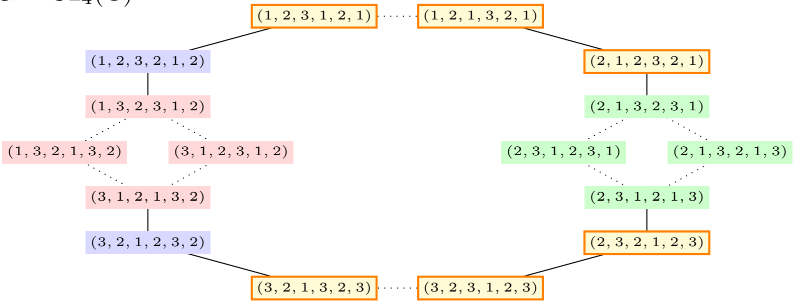
  Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \simeq \mathrm{GC}(\lambda)$.
- $\mathbf{i} = (2,1,3,2,3,1)$. Then

$$\mathrm{ind}_A(2,1,3,2,3,1) = 1, \quad \mathrm{ind}_D(2,1,3,2,3,1) = 1.$$

  Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \not\simeq \mathrm{GC}(\lambda)$. Indeed,

$$\# \text{ of facets of } \Delta_{\mathbf{i}}(\lambda) = 13 \neq 12.$$

$G = \mathrm{SL}_4(\mathbb{C})$



As one may see, the combinatorics of string polytopes heavily depend on that of wiring diagrams. By analyzing wiring diagrams, [Cho–Kim–L, 20^+] enumerates the number $\mathrm{gc}(n)$ of Gelfand–Cetlin type reduced words for $G = \mathrm{SL}_{n+1}(\mathbb{C})$.

n	0	1	2	3	4	5	6	7
$\mathrm{gc}(n)$	1	1	2	6	40	916	102176	68464624
# of all reduced decomp.	1	1	2	16	768	292864	1100742656	48608795688960

Table: The first few terms of $\mathrm{gc}(n)$ (cf. OEIS A337699).

Reduced decompositions having small indices

Definition

A reduced decomposition $\mathbf{i}$ of the longest element in $\mathfrak{S}_{n+1}$ has small indices if there exists a sequence $(\sigma_1, \dots, \sigma_n) \in \{A, D\}^n$ such that

$$\begin{aligned} &\mathrm{ind}_{\sigma_n}(\mathbf{i}) \le \kappa(\sigma_{n-1}, \sigma_n), \\ &\mathrm{ind}_{\sigma_k}\left(C_{\sigma_{k+1}} \circ \cdots \circ C_{\sigma_n}(\mathbf{i})\right) = 0 \quad \text{for all } k = n-1, \dots, 1, \end{aligned}$$

where $\kappa(\sigma_{n-1}, \sigma_n)$ is 2 if $\sigma_{n-1} = \sigma_n$; $n-1$ otherwise.

For example, $\mathbf{i} = (1,3,2,1,3,2)$ has small indices. Take $(D,D,D) \in \{A,D\}^3$. Then,

$$\begin{aligned} &\mathrm{ind}_D(1,3,2,1,3,2) = 2, \\ &C_D(\mathbf{i}) = (1,2,1,) \rightsquigarrow \mathrm{ind}_D(1,2,1) = 0, \\ &C_D(1,2,1) = (1) \rightsquigarrow \mathrm{ind}_D(1) = 0. \end{aligned}$$

All reduced decompositions of the longest element in $\mathfrak{S}_4$ have small indices.

Theorem [Cho–Kim–L–Park 2, 19[+]]

Let $\mathbf{i}$ be a reduced decomposition of the longest element in $\mathfrak{S}_{n+1}$. Let λ be a regular dominant integral weight. If $\mathbf{i}$ has small indices, then $X_{\Delta_{\mathbf{i}}(\lambda)}$ admits a small toric desingularization $\widetilde{X}$. Moreover, $\widetilde{X}$ is obtained by blowing-ups of a Bott manifold.

Corollary

Suppose that $\mathbf{i}$ has small indices. Then, the following holds.

1. $\Delta_{\mathbf{i}}(\lambda)$ is integral for any dominant integral weight λ.
2. For a parabolic subgroup P, $\Delta_{\mathbf{i}}(\lambda_P)$ is reflexive, where λ_P is the weight corresponding to the anticanonical line bundle of G/P.
3. One can compute the Floer theorectical disk potential defined by Fukaya–Oh–Ohta–Ono of the Lagrangian submanifold in G/B given by $\Delta_{\mathbf{i}}(\lambda)$ for any regular dominant integral weight λ.

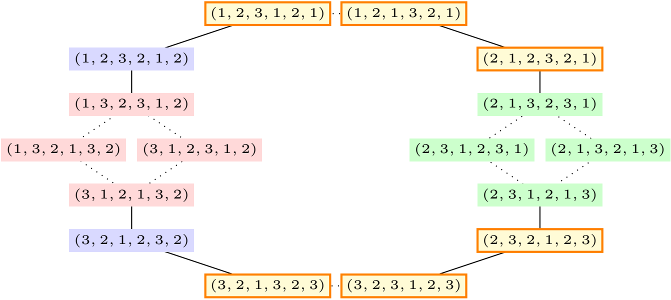
Future works

1. Studying which topological/geometric data can be obtained from different $\Delta_{\mathbf{i}}(\lambda)$.
   - The combinatorial relations among string polytopes (Berenstein–Zelevinsky, 01) and other Newton–Okounkov bodies of G/B (Fujita–Higashitani, 20^+) have been studied.

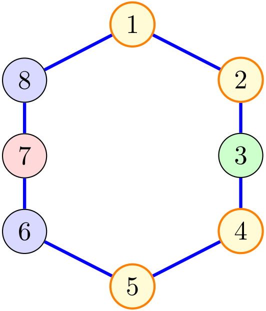
$$\begin{tikzcd}
& G/B \arrow[dl, squiggly, dash, "\text{toric degeneration}"'] \arrow[dr, squiggly, dash, "\text{toric degeneration}"] & \\
X_{\Delta_{\mathbf{i}}(\lambda)} \arrow[d] & & X_{\Delta_{\mathbf{i}'}(\lambda)} \arrow[d] \\
\Delta_{\mathbf{i}}(\lambda) \arrow[rr, dashed, dash, "\text{mutations}"'] & & \Delta_{\mathbf{i}'}(\lambda)
\end{tikzcd}$$

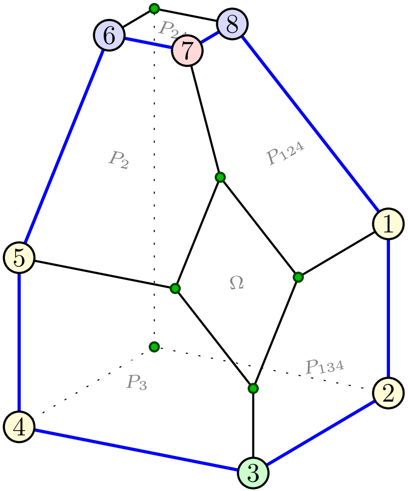
There is an open embedding $U^-_{w_0} \hookrightarrow G/B$ and the the unipotent cell $U^-_{w_0}$ admits a cluster algebra structure. [Fujita–Oya, 20^+] constructed $\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}})$ for each seed $\mathbf{s}$ and proved that $\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}}) \simeq \Delta_{\mathbf{i}}(\lambda)$ when $\mathbf{s}$ comes from $\mathbf{i}$.

$G = \mathsf{SL}_4(\mathbb{C})$



$G = \mathsf{SL}_4(\mathbb{C})$





Question (work in progress)

Describe $\Delta(G/B, \mathcal{L}_\lambda, \nu_{\mathbf{s}})$ for various seeds $\mathbf{s}$ explicitly.

Future works

2. Constructing a completely integrable system associated to $\Delta_{\mathbf{i}}(\lambda)$.
    - For the Gelfand–Cetlin polytope, [Guillemin–Sternberg, 83] provided a completely integrable system. Using this, a detailed description of topology of Gelfand–Cetlin fibers has been studied by [Cho–Kim–Oh, 20].
    - [Harada–Kaveh, 15] proved the *existence* of completely integrable system. However, we don't know the explicit description yet.
3. Generalizing the previous result to other Lie types.
    - String polytopes (and also Newton–Okounkov bodies) are defined for any semisimple Lie groups. We studied the combinatorics of string polytope only for Lie type A.

Thank you for your attention!