On the secondary cohomology of moment-angle complexes (j.w. A.Bahri, T.Panov, J.Song, and D.Stanley)

Ivan Limonchenko
University of Toronto
ilimonch@math.toronto.edu

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Motivation: Topological Data Analysis

For this part of the talk we fix a field of coefficients $\mathbf{k}$.

Definition: Rips complex

Let X be a finite pseudometric space and $U(\varepsilon)$ be its open cover by balls of radius $\varepsilon > 0$ centered at the points of X.
We call the flagification of the nerve of $U(\varepsilon)$ the Rips complex K^ε of X for the radius ε.

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If a discrete set of times $\varepsilon_1, \ldots, \varepsilon_r, \ldots$ is taken into account, this yields a filtration of the corresponding Rips complexes :

$$K^0 = X \subset K^1 \subset K^2 \subset \ldots,$$

where $K^r := K^{\varepsilon_r}$ is a subcomplex in K^{r+1} for all $r \geq 0$.

Persistent homology

Let $\mathcal{K} := (K^r)_{r\geq 0}$. The persistent complex of $\mathcal{K}$ is the sequence of simplicial chain complexes $\mathcal{C} := (C_*^r)_{r\geq 0}$ with injective chain maps $x\colon C_*^r \to C_*^{r+1}$.

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Definition

For $0 \leq r < s$, the (r,s)-persistent homology $PH_*^{r\to s}(\mathcal{K})$ is defined by

$$PH_*^{r\to s}(\mathcal{K}) := \mathrm{im}(x_*^{r\to s}\colon H_*(C_*^r) \to H_*(C_*^s)),$$

where $x^{r\to s}\colon C_*^r \to C_*^s$ is the $(s-r)$-many compositions of x.

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A homology class $\alpha \in H_*(C_*^r)$ is said to

1. be born at K^r if $\alpha \notin PH_*^{t\to r}(\mathcal{K})$ for all $t < r$,
2. die entering K^s if $x_*^{r\to s-1}(\alpha)$ is non-zero in $H_*(C_*^{s-1})$ and $x_*^{r\to s}(\alpha)$ vanishes in $H_*(C_*^s)$.

In this case, the interval $[r,s)$ is called the persistence of $\alpha \in H_*(C_*^r)$.

Barcode

Note that $\mathcal{C} = (C_*^r)_{r \geq 0}$ is a free $\mathbf{k}[x]$-module induced from the injective chain maps. Hence, $PH_*(\mathcal{K}) := \bigoplus_{r<s} PH_*^{r \to s}(\mathcal{K})$ is a $\mathbf{k}[x]$-module and here is the algebraic interpretation of the persistence of a homology class $\alpha \in H_*(C_*^r; \mathbf{k})$.

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Definition

There is an isomorphism of f.g. modules over $\mathbf{k}[x]$:

$$PH_*(\mathcal{C}) \cong \left(\bigoplus_a t^a \cdot \mathbf{k}[x] \right) \oplus \left(\bigoplus_{b,c} t^b \cdot (\mathbf{k}[x]/t^c \mathbf{k}[x]) \right).$$

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The barcode is the set of intervals $\{[a; \infty), [b, b+c)\}$.

The free part above represents the homology generators which are born at K^a and persist for all the future parameters. The torsion part above corresponds to the homology generators whose persistences are $[b, b+c)$.

Observe that for a filtration of Rips complexes

$$K^0 = X \subset K^1 \subset K^2 \subset \ldots,$$

one has:

$$H^{-i,2j}(\mathcal{Z}_{K^r}) \cong \bigoplus_{\substack{J\subset[m]\\|J|=j}} \widetilde{H}_{j-i-1}(K^r_J), \quad \text{for each} \quad r \geq 0$$

and the following sequence of mappings:

$$\mathcal{Z}_{K^1} \to \mathcal{Z}_{K^2} \to \cdots \to \mathcal{Z}_{K^n} \to \cdots$$

induced by the embeddings of simplicial complexes $K^r \hookrightarrow K^{r+1}$.

Bigraded barcode

Set $H^*(\mathcal{Z}_\mathcal{K}) := \bigoplus H_*(\mathcal{Z}_{K^r}) \cong \bigoplus \widetilde{H}_*(K^r_J)$.

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We define the bigraded barcode $BB(X)$ for the Rips filtration complex $\mathcal{K}$ of a pseudometric space X to be the set of intervals: $\{[A;\infty), [B, B+C)\}$.

For each dimension d one gets a barcode $BB_d(X)$ if one considers the part of $H^*(\mathcal{Z}_\mathcal{K})$ consisting of simplicial homology classes of dimension d.

Cohomology of $\mathcal{Z}_K$ and the doubling operation

The bigraded barcode $BB_d(X)$ has certain nice properties, however only a weaker version of the so called *stability theorem* holds for it. One of the reasons is that ordinary (co)homology of a moment-angle complex is **not** invariant under the doubling of a vertex operation.

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Definition

Suppose K is a simplicial complex on $[m] = \{1, 2, \dots, m\}$, $\{v\} \in K$. We call the double of K at v the simplicial complex K^v on $[m] \sqcup w$, whose elements are simplices of K and for each $v \in \sigma \in K$: $\sigma \cup w$ and $(\sigma - v) \cup w$ are also in K^v.

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Note that if $X' = X \sqcup x'$ and $d(x, x') = 0$ for some $x \in X$, then a Rips complex K' of X' is a flagification of K^x.

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Note that if $X' = X \sqcup x'$ and $d(x, x') = 0$ for some $x \in X$, then a Rips complex K' of X' is a flagification of K^x.

Now we are going to introduce the secondary cohomology $HH^*(\mathcal{Z}_K)$ of $\mathcal{Z}_K$, which turns out to be invariant under the doubling operation.

Secondary cohomology: geometric description

Recall that $H_p(\mathcal{Z}_K) \cong \sum_{I\subset[m]} \widetilde{H}_{p-|I|-1}(K_I)$.
Given $j \in [m] \setminus I$, define the homomorphism

$$\phi_{p;I,j}\colon \widetilde{H}_p(K_I) \to \widetilde{H}_p(K_{I\cup\{j\}})$$

induced by the inclusion $K_I \hookrightarrow K_{I\cup\{j\}}$.

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Now define

$$\partial'_p = (-1)^{p+1} \sum_{I \subset [m],\, j \in [m] \setminus I} \varepsilon(j, I)\, \phi_{p;I,j},$$

where $\varepsilon(j, I) = (-1)^{\#\{i \in I \colon i < j\}}$. One can check that $(\partial'_p)^2 = 0$.

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where $\varepsilon(j,I) = (-1)^{\#\{i\in I\colon\, i<j\}}$. One can check that $(\partial'_p)^2 = 0$.
The extra sign $(-1)^{p+1}$ is chosen so that ∂' together with the simplicial boundary ∂ satisfy the bicomplex relation $\partial\partial' = -\partial'\partial$.

For the cohomological version, given $i \in I$, define

$$\psi_{p;i,I} \colon \widetilde{H}^p(K_I) \to \widetilde{H}^p(K_{I\setminus\{i\}})$$

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We define $d'\colon H^*(\mathcal{Z}_K) \to H^*(\mathcal{Z}_K)$ using the decomposition $H^*(\mathcal{Z}_K) = \bigoplus_{I\subset[m]} \widetilde{H}^*(K_I)$; it acts in the bigraded cohomology of $\mathcal{Z}_K$ as follows:

$$d'\colon H^{-k,2l}(\mathcal{Z}_K) \to H^{-k+1,2l-2}(\mathcal{Z}_K).$$

Definition

The (bigraded) secondary cohomology of $\mathcal{Z}_K$ is

$$HH^*(\mathcal{Z}_K) = H\big(H^*(\mathcal{Z}_K), d'\big).$$

Secondary cohomology: algebraic description

Recall that $H^*(\mathcal{Z}_K) \cong H^*[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K], d]$, where $d(u_i) = v_i, d(v_i) = 0$.

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We introduce the second differential d' of bidegree $(1, -2)$ on the Koszul bigraded ring $\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K]$ by setting

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and extending by the Leibniz rule.
Explicitly, the differential d' is defined on square-free monomials u_Jv_I by

$$d'(u_Jv_I)=\sum_{j\in J}\varepsilon(j,J)u_{J\setminus\{j\}}v_I,\quad d'(v_I)=0.$$

It turns out that with d and d' defined above, $\big(\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K], d, d'\big)$ is a bicomplex, that is, d and d' satisfy $dd' = -d'd$.

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Theorem

The secondary cohomology $HH^*(\mathcal{Z}_K)$ is isomorphic to the first double cohomology of the bicomplex $\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K],d,d'\big)$:

$$HH^*(\mathcal{Z}_K)\cong H\big(H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K],d\big),d'\big).$$

It follows that $HH(\mathcal{Z}_K)$ is a combinatorial invariant of K.

Secondary cohomology and the doubling operation

Theorem

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The idea behind the proof is to show that the chain map $\phi\colon CH(\mathcal{Z}_{K^v}) \cong \oplus_{I\subset[m]\cup w}\widetilde{H}(K^v_I) \to CH(\mathcal{Z}_K) \cong \oplus_{I\subset[m]}\widetilde{H}(K_I)$, determined by the property that for $\alpha \in \widetilde{H}(K^v_I)$

$$\phi(\alpha) = \begin{cases} 0 & w \in I \\ \alpha & w \notin I \end{cases}$$

is a weak equivalence.

Secondary cohomology and the doubling operation

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is a weak equivalence.

For $HH(\mathcal{Z}_K)$ we are able to define the barcode and show the stability theorem holds for it.

First examples

The $(m-1)$-simplex

For a simplicial complex K without ghost vertices, the following statements are equivalent:

(a) all full subcomplexes of K are acyclic;

(b) $K = \Delta^{m-1}$ and $\mathcal{Z}_K = (D^2)^m$;

(c) $\mathcal{Z}_K$ is acyclic;

(d) $HH^*(\mathcal{Z}_K) \cong \mathbb{Z}$.

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The boundary of an $(m-1)$-simplex

Let $K = \partial\Delta^{m-1}$. The only non-trivial cohomology groups are

$$H^0(\mathcal{Z}_K) = H^{0,0}(\mathcal{Z}_K) \cong \mathbb{Z}\langle[1]\rangle,$$

$$H^{2m-1}(\mathcal{Z}_K) = H^{-1,2m}(\mathcal{Z}_K) \cong \mathbb{Z}\langle[u_1 v_2 v_3 \cdots v_m]\rangle$$

The differential $d' = 0$ on $H^*(\mathcal{Z}_K)$, and $HH^*(\mathcal{Z}_K) = H^*(\mathcal{Z}_K)$.

The boundary of a 4-gon

Let K be the boundary of a square labeled clockwisely.
Then $\mathcal{Z}_K \cong S^3 \times S^3$ and its non-trivial cohomology groups are

$$H^0(\mathcal{Z}_K) = H^{0,0}(\mathcal{Z}_K) \cong \mathbb{Z}\langle 1\rangle,$$
$$H^3(\mathcal{Z}_K) = H^{-1,4}(\mathcal{Z}_K) \cong \mathbb{Z}\langle [u_1 v_3], [u_2 v_4]\rangle,$$
$$H^6(\mathcal{Z}_K) = H^{-2,8}(\mathcal{Z}_K) \cong \mathbb{Z}\langle [u_1 u_2 v_3 v_4]\rangle.$$

The differential $d' = 0$ on $H^*(\mathcal{Z}_K)$, and we have

$$HH^*(\mathcal{Z}_K) = H^*(\mathcal{Z}_K)$$

in this case.

The boundary of a 5-gon

Let K be the boundary of a pentagon labeled clockwisely.
Then $\mathcal{Z}_K \cong (S^3 \times S^4)^{\#5}$ and its non-trivial cohomology groups are

$$H^0(\mathcal{Z}_K) = H^{0,0}(\mathcal{Z}_K) \cong \mathbb{Z}\langle 1\rangle,$$
$$H^3(\mathcal{Z}_K) = H^{-1,4}(\mathcal{Z}_K) \cong \mathbb{Z}\langle [u_1 v_3], [u_1 v_4], [u_2 v_4], [u_2 v_5], [u_3 v_5]\rangle,$$
$$H^4(\mathcal{Z}_K) = H^{-2,6}(\mathcal{Z}_K) \cong \mathbb{Z}\langle [u_4 u_5 v_2], [u_2 u_3 v_5], [u_5 u_1 v_3], [u_3 u_4 v_1], [u_1 u_2 v_4]\rangle$$
$$H^7(\mathcal{Z}_K) = H^{-3,10}(\mathcal{Z}_K) \cong \mathbb{Z}\langle [u_1 u_2 u_3 v_4 v_5]\rangle.$$

There is only one non-trivial differential d':

$$0 \longrightarrow H^{-2,6}(\mathcal{Z}_K) \xrightarrow{d'} H^{-1,4}(\mathcal{Z}_K) \longrightarrow 0$$

The boundary of a 5-gon

It is given on the basis elements by

$$\begin{aligned}
d'[u_4 u_5 v_2] &= [u_5 v_2] - [u_4 v_2] = [u_2 v_5] - [u_2 v_4],\\
d'[u_2 u_3 v_5] &= [u_3 v_5] - [u_2 v_5],\\
d'[u_5 u_1 v_3] &= [u_1 v_3] - [u_5 v_3] = [u_1 v_3] - [u_3 v_5],\\
d'[u_3 u_4 v_1] &= [u_4 v_1] - [u_3 v_1] = [u_1 v_4] - [u_1 v_3],\\
d'[u_1 u_2 v_4] &= [u_2 v_4] - [u_1 v_4].
\end{aligned}$$

The corresponding matrix

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

has rank 4 and defines a homomorphism onto a direct summand of $\mathbb{Z}^5$.

More examples

It follows that the non-trivial secondary cohomology groups for a pentagon K are

$$HH^{0,0}(\mathcal{Z}_K) \cong HH^{-1,4}(\mathcal{Z}_K) \cong HH^{-2,6}(\mathcal{Z}_K) \cong HH^{-3,10}(\mathcal{Z}_K) \cong \mathbb{Z}.$$

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The next example shows that $HH^*(\mathcal{Z}_K)$ may have torsion.

The minimal triangulation of $\mathbb{R}P^2$

Let K be the minimal triangulation of $\mathbb{R}P^2$ with 6 vertices. In this case, $H^{-3,12}(\mathcal{Z}_K) = \mathbb{Z}_2$, $H^{-3,10}(\mathcal{Z}_K) = \mathbb{Z}^6$, $H^{-2,8}(\mathcal{Z}_K) = \mathbb{Z}^{15}$, $H^{-1,6}(\mathcal{Z}_K) = \mathbb{Z}^{10}$, $H^{0,0}(\mathcal{Z}_K) = \mathbb{Z}$ and all other bigraded cohomology groups vanish.
Hence, for the differential d' we have

$$0 \xrightarrow{d'} H^{-3,12}(\mathcal{Z}_K) \xrightarrow{d'} 0,$$

$$0 \xrightarrow{d'} H^{-3,10}(\mathcal{Z}_K) \xrightarrow{d'} H^{-2,8}(\mathcal{Z}_K) \xrightarrow{d'} H^{-1,6}(\mathcal{Z}_K) \xrightarrow{d'} 0.$$

The first sequence implies that $HH^{-3,12}(\mathcal{Z}_K) = \mathbb{Z}_2$.

General results

Theorem

Let $K' = K \sqcup \{w\}$ for a simplicial complex K. Then,

$$HH^{-k,2\ell}(\mathcal{Z}_{K'}) = \begin{cases} \mathbb{Z} & (-k,2\ell) = (0,0),\ (-1,4); \\ 0 & \text{otherwise.} \end{cases}$$

This gives $HH(\mathcal{Z}_K)$ for $K = \{3\,\mathrm{pts}\}$, $\mathcal{Z}_K \simeq (S^3)^{\vee 3} \vee (S^4)^{\vee 2}$.

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Theorem

Let K be a simplicial complex on $[m]$ s.t. $K_{[s]} = \Delta^{s-1}$. Let K' be the simplicial complex on $[m+1]$, where we have glued an s-simplex to K by identifying a facet of the simplex with $K_{[s]}$. Then one has

$$HH(\mathcal{Z}_{K'}) \cong HH(\mathcal{Z}_L),$$

where $L = K_{[m]-[s]} \sqcup w$.

The last two statements imply the next result.

Theorem (first part)

For a simplicial complex K without a ghost vertex, the following statements are equivalent:

(a) all full subcomplexes of K are homotopy discrete sets of points;

(b) K is flag and its 1-skeleton $\mathrm{sk}^1(K)$ is a chordal graph;

(c) K can be obtained by iterating the procedure of attaching a simplex along a face, starting from a simplex. Here, by attaching along $\varnothing$ we mean adding a disjoint simplex.

Theorem (second part)

Moreover, when K is a flag complex, (a), (b), and (c) are equivalent to

(d) $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;

(e) K is a Golod complex.

Finally, each of the (a), (b), and (c) implies

(f) Either K is a simplex, or $HH(\mathcal{Z}_K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

THANK YOU FOR YOUR ATTENTION!