Stability of Schubert classes and Bott-Samelson resolutions

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Flag varieties and Schubert varieties

$G_n := GL_n(\mathbb{C})$

$B_n :=$ Borel subgp of G_n (upper triangular matrices)

$B_n^- :=$ opp. Borel (lower triang)

$Fl_n := G_n/B_n$ the flag variety

Schubert variety $X_w^{(n)} = \overline{B_n^- w B_n}$ $\quad B^-$orbit closure of $w \in S_n$

$\cdots$ irreducible, codim $\ell(w)$, *at worst rational singularities*

Schubert classes $\sigma_w^{(n)} = [X_w^{(n)}]$ form a basis of the Chow ring $A^*(Fl_n)$.

Schubert Calculus

Define the coefficients $c_{uv}^w \in \mathbb{Z}$ $(u, v, w \in S_n)$ by

$$\sigma_u^{(n)} \sigma_v^{(n)} = \sum_{w \in S_n} c_{uv}^w \sigma_w^{(n)}$$

Kleiman's transversality theorem implies that $c_{uv}^w \in \mathbb{Z}_{\geq 0}$

$\Rightarrow$ **Goal**. Find a nice "combinatorial" formula for c_{wv}^u.

Stability of Schubert classes

$Fl_n \cong$ the space of flags $U_\bullet : U_1 \subset \cdots \subset U_{n-1} \subset \mathbb{C}^n$

$\Rightarrow \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_{n-1}$ tautological bundle bundles on Fl_n

$$A^*(Fl_n) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle e_i(x_1, \ldots, x_n), i = 1, \ldots, n\rangle}, \quad c_1((\mathcal{U}_i/\mathcal{U}_{i-1})^\vee) \mapsto x_i$$

$\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ (first n coordinates) induces an embedding $f_n : Fl_n \hookrightarrow Fl_{n+1}$ and $S_n \hookrightarrow S_{n+1}$

$$f_n^* : A^*(Fl_{n+1}) \twoheadrightarrow A^*(Fl_n) \qquad (\text{ set } x_{n+1} = 0 \text{ })$$

$\Rightarrow$ The direct limit of $A^*(Fl_n)$ contains the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$.

Fact (Stability of Schubert class).

$$f_n^*(\sigma_w^{(n+1)}) = \sigma_w^{(n)}, \quad w \in S_n$$

$\Rightarrow \exists!$ a polynomial representative of the limit $\sigma_w^{(\infty)}$, the Schubert polynomial $\mathfrak{S}_w(x)$

$$\mathfrak{S}_w \cdot \mathfrak{S}_v = \sum_{u \in S_\infty} c_{wv}^u \mathfrak{S}_u \quad \text{in } \mathbb{Z}[x_1, x_2, \ldots]$$

Algebraic Cobordism Ω^*

Oriented cohomology theory A^*

- A functor: X (smooth "manifolds") $\mapsto A^*(X)$ (graded rings)
- pushforward, projective bundle formula, extended homotopy property
- Chern classes
- Fundamental classes of "submanifolds with mild singularities (l.c.i)"

e.g.

Chow ring $\mathrm{CH}^*(X)$, (connective) K-theory $\mathrm{CK}^*(X)$

Algebraic Cobordism $\Omega^*(X)$ by Levine-Morel is the universal one

Chern classes, Formal group law, Lazard ring

Line bundles L, M over X, $\quad F_A(u,v) = u + v + (\text{ higher degree }) \in A^*(\mathrm{pt})[[u,v]]$.

$$c_1^A(L \otimes M) = F_A(c_1(L), c_1(M))$$

There is $\chi_A(u) \in A^*(\mathrm{pt})[[u]]$ such that $F_A(u, \chi(u)) = 1 \quad \Rightarrow \quad c_1^A(L^\vee) = \chi_A(c_1^A(L))$.

Example

- Chow ring $\qquad F_{\mathrm{CH}}(u,v) = u + v, \quad \chi_{\mathrm{CH}}(u) = -u$
- K-theory $\qquad F_K(u,v) = u + v - uv, \quad \chi_K(u) = -u/(1-u)$

Algebraic Cobordism Ω^*

$\Omega^*(\mathrm{pt}) = \mathbb{L}$ (Lazard ring) $\cong$ a polynomial ring with infinite variables

$c_1^\Omega(L \otimes M) = F_\Omega(c_1(L), c_1(M)) = c_1^\Omega(L) + c_1^\Omega(M) + \sum_{i,j} c_{i,j} c_1^\Omega(L)^i c_1^\Omega(M)^j \quad (c_{i,j} \in \mathbb{L})$

Cobordism ring of flag varieties

Theorem (Hornbostel–Kiritchenko).

$$\Omega^*(Fl_n) \cong \mathbb{L}[x_1, \ldots, x_n]/\mathbb{S}_n, \qquad c_1((\mathcal{U}_i/\mathcal{U}_{i-1})^\vee) \mapsto x_i$$

where $\mathbb{S}_n$ is the ideal of symmetric polynomials in $x_1, \ldots, x_n$ in positive degree.

$i_n : Fl_n \to Fl_{n+1}$ induces $i_n^* : \Omega^*(Fl_{n+1}) \to \Omega^*(Fl_n)$ $\quad (\cdots$ set $x_{n+1} = 0)$

$$\varprojlim \Omega^*(Fl_n) \cong \mathbb{L}[[x_1, x_2, x_3, \ldots]]_{bd}/\mathbb{S}_\infty$$

"Schubert classes" in cobordism

A Schubert variety $X_w^{(n)}$ has at worst rational singularities (up to K-theory it behaves nice)

$\Rightarrow$ $X_w^{(n)}$ may not be a local complete intersection

$\Rightarrow$ the class of $X_w^{(n)}$ in $\Omega^*(Fl_n)$ may not be well-defined

Problem

What can replace the Schubert classes in Ω^*?

An answer:

The class of a resolution of $X_w^{(n)}$ in $\Omega^*(Fl_n)$ could be a replacement of the Schubert class

$\Rightarrow$ We can consider Bott–Samelson resolutions (*Bott–Samelson classes in* $\Omega^*(Fl_n)$)

Problem

Do Bott–Samelson classes have the stability?

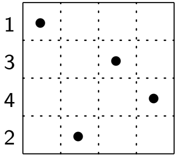
An answer: Yes. We can consider the limit class in $\varprojlim \Omega^*(Fl_n)$.

Schubert conditions (essential ones)

Diagram of $w \in S_n$ and essential boxes

1. Consider the transpose of the permutation matrix of w and place $\bullet$ in the position of 1.
2.
3.
4.
5.

e.g $w = 1342$



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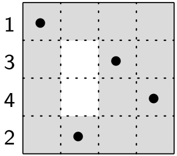

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1. Consider the transpose of the permutation matrix of w and place $\bullet$ in the position of 1.
2. Delete the boxes on the right and below of each $\bullet$.
3. Call the remaining boxes the diagram of w, and denote by $D(w)$
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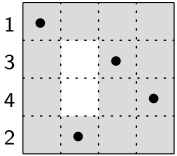

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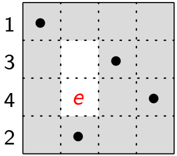
4. The south east corners of $D(w)$ are essential boxes $(p_1, q_1), \ldots, (p_s, q_s)$
5.

e.g $w = 1342$



$(p_1, q_1) = (3, 2)$

Schubert conditions (essential ones)

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5. Let r_i be the number of bullets in the northwest of (p_i, q_i).

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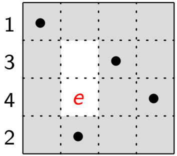

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$r_1 = 1$

Schubert conditions (essential ones)

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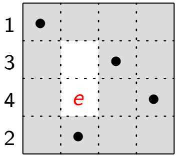
Fact: Let $F_i = \langle e_{n-i+1}, \ldots, e_n \rangle$ ($F_\bullet$ the reference flag). Then we have

$$U_\bullet \in X_w \quad \Leftrightarrow \quad \dim(U_{p_i}, E/F_{n-q_i}) \geq p_i - r_i, \ \forall i = 1, \ldots, s.$$

e.g $w = 1342$



$(p_1, q_1) = (3, 2)$

$r_1 = 1$

$U_\bullet \in X_{1342} \Leftrightarrow \dim(U_3 \cap F_2) \geq 2$

Bott–Samelson Resolutions

Bott–Samelson variety

$v \in S_n$ with $r := \ell(v)$ $\qquad \underline{v} = s_{i_1} s_{i_2} \cdots s_{i_r}$ $\quad$ reduced word

P_i: i-th minimal parabolic subgroup in G_n

$$BS(\underline{v}) = w_0 P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_r}/B$$

$BS(\underline{v})$ is smooth and has dimension r.

Magyar's description by fiber product [Magyar 1998]

$$\begin{aligned} BS(\underline{v}) &\cong \{F_\bullet\} \times^{G/P_{i_1}} Fl_n \times^{G/P_{i_2}} \cdots \times^{G/P_{i_r}} Fl_n \\ &\quad \big([g_1, \dots, g_r] \mapsto ([w_0], [g_1], [g_1 g_2], \dots, [g_1 \cdots g_r])\big) \\ &= \{(U_\bullet^{[0]}, \dots, U_\bullet^{[r]}) \in (Fl_n)^{r+1} \mid U_i^{[k-1]} = U_i^{[k]}, i \neq i_k, \forall k\} \quad (U_\bullet^{[0]} := F_\bullet) \end{aligned}$$

Bott-Samelson Resolution of Schubert Variety

Let $w \in S_n$, $w_0^{(n)}$ the longest in S_n. Bott–Samelson resolution of X_w is

$$BS(\underline{w_0^{(n)} w}) \to X_w \qquad (U_\bullet^{[0]}, \dots, U_\bullet^{[r]}) \mapsto U_\bullet^{[r]}$$

where $r := \ell(w_0^{(n)} w)$.

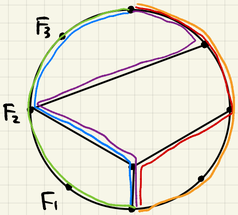
Example

$$w = 1342 = s_2 s_3, \quad w_0^{(4)} = s_1 s_2 s_3 s_1 s_2 s_1, \quad \underline{w_0^{(4)} w} = s_1 s_3 s_2 s_1$$

$$BS(s_1 s_3 s_2 s_1) \cong \{F_\bullet\} \times^{G/P_1} Fl_4 \times^{G/P_3} Fl_4 \times^{G/P_2} Fl_4 \times^{G/P_1} Fl_4$$

$$\begin{array}{ccccccccc}
\boxed{\begin{array}{c} F_3 \\ F_2 \\ F_1 \end{array}} & \begin{array}{c} = \\ = \\ \end{array} & \boxed{\begin{array}{c} U_3^{[1]} \\ U_2^{[1]} \\ U_1^{[1]} \end{array}} & \begin{array}{c} \\ = \\ = \end{array} & \boxed{\begin{array}{c} U_3^{[2]} \\ U_2^{[2]} \\ U_1^{[2]} \end{array}} & \begin{array}{c} = \\ \\ = \end{array} & \boxed{\begin{array}{c} U_3^{[3]} \\ U_2^{[3]} \\ U_1^{[3]} \end{array}} & \begin{array}{c} = \\ = \\ \end{array} & \boxed{\begin{array}{c} U_3^{[4]} \\ U_2^{[4]} \\ U_1^{[4]} \end{array}}
\end{array}$$

Elnitsky's Rhombic Tiling [Elnitsky 1997]



$BS(s_1 s_3 s_2 s_1) \longrightarrow X_{1342}$

$(F_\bullet, U_\bullet^{[1]}, \dots, U_\bullet^{[4]}) \mapsto U_\bullet^{[4]}$

$\boxed{U_1^{[1]} \text{ is the only extra info in } BS}$

Stability of BS resolutions

Theorem (Hudson–M.–Perrin)

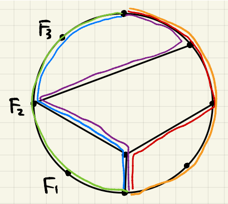
Let $w \in S_n$ and fix $\underline{w_0^{(n)}w}$. Set $\underline{w_0^{(n+1)}w} := s_1 \cdots s_n \underline{w_0^{(n)}w}$.

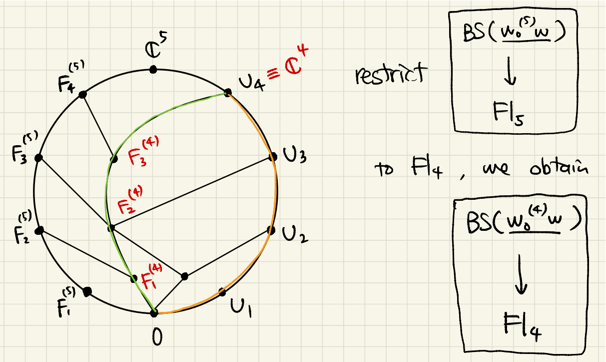
Then the natural embedding $f_n : Fl_n \to Fl_{n+1}$ induces the following fiber diagram

$$\begin{tikzcd} BS(\underline{w_0^{(n)}w}) \arrow[r] \arrow[d] & BS(\underline{w_0^{(n+1)}w}) \arrow[d] \\ Fl_n \arrow[r] & Fl_{n+1} \end{tikzcd}$$

Example

$$w = 1342, \qquad \underline{w_0^{(4)}}\, w = s_1 s_3 s_2 s_1, \qquad \underline{w_0^{(5)}}\, w = s_1 s_2 s_3 s_4 \cdot s_1 s_3 s_2 s_1$$





“Schubert classes” in cobordism

$[BS(\underline{w_0^{(n)}w}) \to Fl_n]$ the class of a BS resolution of X_w in $\Omega^*(Fl_n)$

Theorem (Hudson–M.–Perrin)

Let $w \in S_n$ and fix $\underline{w_0^{(n)}w}$.

Set $\underline{w_0^{(m+1)}w} = s_1 s_2 \cdots s_m \underline{w_0^{(m)}w}$ inductively for all $m \geq n$.

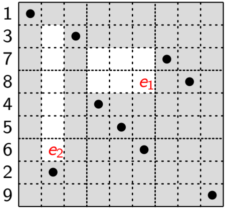
- $f_m^*[BS(\underline{w_0^{(m+1)}w}) \to Fl(\mathbb{C}^{m+1})] = [BS(\underline{w_0^{(m)}w}) \to Fl(\mathbb{C}^m)]$
- There is a corresponding unique element $\mathcal{S}_w(x)$ in the limit $\mathbb{L}[[x_1, x_2, x_3, \ldots]]_{bd}/\mathbb{S}_\infty$

-Note $\mathcal{S}_w(x)$ depends on the choice $\underline{v}^{(n)} = \underline{w_0^{(n)}w}$. To emphasize this, we write $\mathcal{S}_w^{\underline{v}^{(n)}}(x)$.

-Hudson–M.–Perrin computed $\mathcal{S}_w(x)$ for dominant permutations w in the case of *infinitesimal cohomology theory*, using divided difference operators.

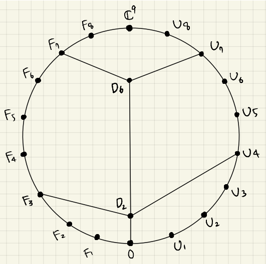
Another type of resolutions in vexillary case

Example. $w = 137845629$



$$(p_1, q_1) = (4, 6), \quad r_1 = 2$$
$$(p_2, q_2) = (7, 2), \quad r_2 = 1$$
$$\begin{cases} \dim(U_4 \cap F_3) \geq 2 \\ \dim(U_7 \cap F_7) \geq 6 \end{cases} \quad \Leftrightarrow \quad U_\bullet \in X_w$$

Define $Z_w := \{(D_2, D_4, U_\bullet) \mid U_\bullet \in X_w,\ D_2 \subset U_4 \cap F_3,\ D_6 \subset U_7 \cap F_7\}$



$Z_w \to X_w$

$(D_2, D_6, U_\bullet) \mapsto U_\bullet$

is a resolution of singularities

$\because$ w is 2143-avoiding (vexillary)

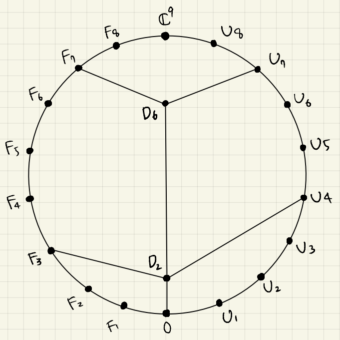
i.e., ess. boxes lined down from NE to SW

The "rhombic tiling" looks like leaf venation

Bott–Samelson type of description of Z_w

As a tower of partial flag varieties

$$\begin{array}{ccccccccc} Gr(2;F_3) & \leftarrow & Gr(4;F_7/D_2) & \longleftarrow & Fl_{\geq 1}(\mathbb{C}^9/D_6) & \longleftarrow & Fl_{\geq 2}(U_7/D_2) & \longleftarrow & Fl(U_4) = Z_w \\ D_2 & & D_6 & & U_7 \subset U_8 & & U_4 \subset U_5 \subset U_6 & & U_1 \subset U_2 \subset U_3 \end{array}$$



$$\{F_\bullet\} \times^{Fl_{\{3,\ldots,8\}}} Fl_{\{2,\ldots,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,6\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,4,\ldots,8\}} \times^{Fl_{\{4,\ldots,8\}}} Fl_9$$

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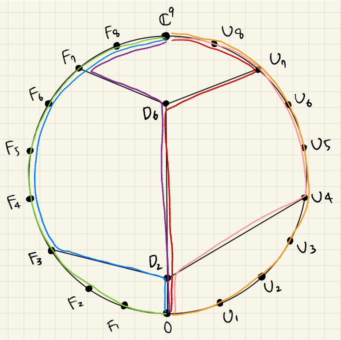

$$\{F_\bullet\} \times^{Fl_{\{3,\dots,8\}}} Fl_{\{2,\dots,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,6\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,4,\dots,8\}} \times^{Fl_{\{4,\dots,8\}}} Fl_9$$

Bott–Samelson type of description of Z_w

$$Gr(2; F_3) \leftarrow Gr(4; F_7/D_2) \leftarrow Fl_{\geq 1}(\mathbb{C}^9/D_6) \leftarrow Fl_{\geq 2}(U_7/D_2) \leftarrow Fl(U_4)$$

$$\{F_\bullet\} \times^{Fl_{\{3,\dots,8\}}} Fl_{\{2,\dots,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,6\}}} Fl_{\{2,6,7,8\}} \times^{Fl_{\{2,7,8\}}} Fl_{\{2,4,\dots,8\}} \times^{Fl_{\{4,\dots,8\}}} Fl_9$$

$$\{F_\bullet\} \times^{G/P(3)} G/P(2) \times^{G/P(2,7)} G/P(2,6) \times^{G/P(2,6,9)} G/P(2,6) \times^{G/P(2,7)} G/P(2,4) \times^{G/P(4)} G/B$$

$$w_0 P(3) \quad \times_{P(2)} \quad P(2,7) \quad \times_{P(2,6)} \quad P(2,6,9) \quad \times_{P(2,6)} \quad P(2,7) \quad \times_{P(2,4)} \quad P(4)/ \quad B$$