

The rational string coproduct and the Hodge decomposition

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Plan and Notations

Plan

1. String topology and string coproduct
2. Hodge decomposition of the free loop space homology
3. Main result and Pure manifolds

Notations

M : closed oriented m -dim manifold

$LM = \text{Map}(S^1, M)$: free loop space ($S^1 = I/\partial I$)

$H_*(-) = H_*(-; \mathbb{K})$: the singular homology over a field \mathbb{K}

$H_*(LM)$: the **loop homology**

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String Topology

String topology : a study of algebraic structures on $H_*(LM)$

The loop homology $H_*(LM)$ has the following algebraic structures;

- commutative graded algebra with unit (Chas-Sullivan)
- Batalin-Vilkovisky algebra (Chas-Sullivan)
- 2-dim TQFT (Cohen-Godin)
- Homological Conformal Field Theory (Godin)
- infinitesimal bialgebra (Sullivan)

Chas-Sullivan Loop Product

$$H_p(LM) \otimes H_q(LM) \longrightarrow H_{p+q-m}(LM) : \text{degree } -m$$

- The loop product is defined by mixing
 - ▶ the intersection product on $H_*(M)$,
 - ▶ the product on $H_*(\Omega M)$ induced by concatenation of loops.
- Homotopy invariant (Cohen-Klein-Sullivan)
- Computations
 - ▶ S^n, CP^n (Cohen-Jones-Yan)
 - ▶ Lie group (Hepworth)
 - ▶ Lens spaces (Lupercio-Urbe-Xicontencatl)
 - ▶ \vdots

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String Coproduct

$$H_*(LM, M) \longrightarrow H_*(LM, M) \otimes H_*(LM, M) : \text{degree } 1 - m$$

- The string coproduct is defined by cutting of loops at self intersection points.
- Homotopy invariant (Hingston-Wahl)
- Computations
 - ▶ S^1 (Basu)
 - ▶ S^n over \mathbb{Q} (N.)

Construction of the string coproduct

$$h : LM \times I \longrightarrow M \times M, \quad h(\gamma, t) := (\gamma(0), \gamma(t))$$

$$P = \{(\gamma, t) \in LM \times I \mid \gamma(0) = \gamma(t)\}$$

$$\text{cut} : P \longrightarrow LM \times LM, \quad \text{cut}(\gamma, t) = (\gamma_{[0,t]}, \gamma_{[t,1]})$$

Consider the following diagram

$$\begin{array}{ccccc} LM \times I & \xleftarrow{\bar{\Delta} : \text{inclusion}} & P & \xrightarrow{\text{cut}} & LM \times LM \\ \downarrow h & & \downarrow & & \\ M \times M & \xleftarrow{\Delta : \text{diagonal}} & M & & \end{array}$$

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Apply the homology functor

$$\begin{array}{ccccc} H_*(LM \times I) & \xleftarrow{\tilde{\Delta}_*} & H_*(P) & \xrightarrow{(\text{cut})_*} & H_*(LM \times LM) \\ \downarrow h_* & & \downarrow & & \\ H_*(M \times M) & \xleftarrow{\Delta_*} & H_*(M) & & \end{array}$$

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Extend the shriek map

$$\begin{array}{ccccc} H_*(LM \times I) & \xrightarrow{\tilde{\Delta}!} & H_*(P) & \xrightarrow{(\text{cut})_*} & H_*(LM \times LM) \\ \downarrow h_* & & \downarrow & & \\ H_*(M \times M) & \xrightarrow[\text{shriek map}]{\Delta!} & H_*(M) & & \end{array}$$

Construction of the string coproduct

The composite

$$H_*(LM \times I) \xrightarrow{(\text{cut})_* \circ \tilde{\Delta}!} H_*(LM \times LM)$$

induces a morphism between the relative homology

$$H_*((LM, M) \times (I, \partial I)) \xrightarrow{(\text{cut})_* \circ \tilde{\Delta}!} H_*((LM, M) \times (LM, M))$$

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$([I, \partial I] \in H_1(I, \partial I) : \text{fundamental class})$

Hodge Decomposition of $H_*(LM; \mathbb{Q})$

M : simply-connected

$$\varphi_n : LM \longrightarrow LM, \quad \gamma(t) \longmapsto \gamma(nt)$$

$$\varphi_{n*} : H_*(LM; \mathbb{Q}) \longrightarrow H_*(LM; \mathbb{Q})$$

Then,

- $1, n, n^2, n^3, \dots, n^k, \dots$: the eigenvalues of φ_{n*}
- $H_*^{(k)} := H_*^{(k)}(LM; \mathbb{Q})$: the eigenspace of n^k
- $H_*(LM; \mathbb{Q}) = \bigoplus_{k \geq 0} H_*^{(k)}$: **Hodge decomposition**

The direct summand $H_*^{(0)}$ satisfies

$$H_*^{(0)} \cong H_*(M; \mathbb{Q})$$

The direct summand $H_*^{(1)}$

Module derivation

- $\rho : S^1 \times LM \longrightarrow LM$: the rotation of loops
- $\text{ev}_0 : LM \longrightarrow M$, $\text{ev}_0(\gamma) = \gamma(0)$

The image of the following composite contains in $H_{(1)}^*$;

$$H^*(M) \xrightarrow{\text{ev}_0^*} H^*(LM) \xrightarrow{\rho^*} H^*(S^1 \times LM) \xrightarrow{/[S^1]} H^{*-1}(LM)$$

The space of sections

- $\text{Sec}(\text{ev}_0)$: the space of sections of ev_0

Then, the evaluation map $M \times \text{Sec}(\text{ev}_0) \rightarrow LM$ induces

$$\pi_i(\text{Sec}(\text{ev}_0)) \otimes \mathbb{Q} \cong H_{i+m}^{(1)} \quad \text{for } i \geq 1 \quad (\text{Félicx-Thomas})$$

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Main Theorem (N.)

M : simply-connected *pure* manifold

$\vee : H_*(LM, M; \mathbb{Q}) \longrightarrow H_*(LM, M; \mathbb{Q})^{\otimes 2} : \text{the string coproduct} / \mathbb{Q}$

(0) If $\dim \pi_{\text{odd}}(M) \otimes \mathbb{Q} - \dim \pi_{\text{even}}(M) \otimes \mathbb{Q} = 0$, then

$$\vee : H_*^{(k)} \longrightarrow \bigoplus_{p+q=k} H_*^{(p)} \otimes H_*^{(q)} : \text{degree } 0$$

(1) If $\dim \pi_{\text{odd}}(M) \otimes \mathbb{Q} - \dim \pi_{\text{even}}(M) \otimes \mathbb{Q} = 1$, then

$$\vee : H_*^{(k)} \longrightarrow \bigoplus_{p+q=k-1} H_*^{(p)} \otimes H_*^{(q)} : \text{degree } -1$$

(2) If $\dim \pi_{\text{odd}}(M) \otimes \mathbb{Q} - \dim \pi_{\text{even}}(M) \otimes \mathbb{Q} \geq 2$, then \vee is trivial.

Sullivan models and Pure manifolds

$\wedge V = (\wedge V, d)$: minimal Sullivan model for M , that is,

- $V = \{V^n\}_{n \geq 2}$: graded \mathbb{Q} -vector space
- $(\wedge V, d)$: free commutative differential graded algebra with a decomposable differential
- $\exists C^*(M; \mathbb{Q}) \longrightarrow \cdots \longleftarrow \wedge V$: a seq of quasi-isom of dg-algebras

Definition

A simply-connn space M is *pure* if the minimal Sullivan model $\wedge V$ satisfies

$$\dim V < \infty, \quad d(V^{\text{even}}) = \{0\}, \quad d(V^{\text{odd}}) \subset \wedge V^{\text{even}}$$

Example

- $S^n, \mathbb{C}P^n$, compact Lie group, homogeneous space
- $H^*(M; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n] / \langle \text{reg seq} \rangle$

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Thank you for listening!