Cohomological rigidity on Fano generalized Bott manifolds

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Generalized Bott manifold

\[ B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \ldots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0, \]

\[ P(\mathbb{C} \oplus \xi_1 \oplus \ldots \oplus \xi_m) \xrightarrow{\mathbb{C}P^n} \{ \text{a point} \} \]

\( B_m : \) m-stage generalized Bott manifold

\[ \Rightarrow B_m : \text{a smooth projective toric variety of dim } N = \frac{m}{\sum_{i=1}^{m} n_i}. \]

\[ \Rightarrow B_m / T^N \cong \Delta^{n_1} \times \Delta^{n_2} \times \ldots \times \Delta^{n_m} \]

Ex). \( n_i = 1 \forall i \Rightarrow B_m \) is a Bott manifold.

\[ \prod_{i=1}^{m} \mathbb{C}P^{n_i} : \text{trivial generalized Bott manifold}. \]
Fano variety

In algebraic geometry, a smooth Fano variety is a smooth projective variety $X$ whose anticanonical divisor $-K_X$ is ample. i.e., the 1st Chern class can be represented by a positive definite differential form.

Ex.) $\mathbb{CP}^n$ $\cup_n$

- smooth hypersurfaces in $\mathbb{CP}^n$ of degree $\leq n$.
- Hirzebruch surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}^a)$, $-1 \leq a \leq 1$. 
Conjecture [Cho-Lee-Masuda-P.] Let $X$ and $Y$ be smooth Fano toric varieties. If there is a $c_1$-preserving cohomology ring isomorphism from $H^*(X;\mathbb{Z})$ to $H^*(Y;\mathbb{Z})$, then $X$ and $Y$ are isomorphic as toric varieties.

Affirmative results


(2) $\dim_0 X \leq 4$ or Picard number $\geq 2\dim_0 X - 2$.
   (Higashitani-Kurimoto-Masuda)
Today

① Characterization of smooth Fano toric varieties using the notion of primitive collections (Batyrev's work)

② Fano generalized Bott manifolds

③ $c_1$-cohomological rigidity for certain Fano generalized Bott manifolds.
Characterization of Fano toric varieties

Let $\Sigma$ be a complete non-singular fan. A subset $P = \{p_1, \ldots, p_k\}$ of $\Sigma(1)$ is a primitive collection if $P$ is not contained in $\sigma(1)$ for all $\sigma \in \Sigma$, but any proper subset is.

Ex.1

Let $u_p$ be the primitive integral vector generating the ray $p$. For each primitive collection $P = \{p_1, \ldots, p_k\}$, $\sum_{i=1}^{k} u_{p_i}$ lies in the relative interior of a cone $\gamma \in \Sigma$.

Define $\deg(P) = k - \sum_{\rho \in \gamma} a_\rho$. 

Note: $\mathbb{R}^n$ is assumed to have the standard basis $\{e_1, e_2, \ldots, e_n\}$, where $e_i$ is the $i$-th standard basis vector.
Theorem [Batyrev]

Let $\Sigma$ be the projective smooth toric variety.
Let $PC(\Sigma)$ be the set of primitive collections of $\Sigma$.
Then the toric variety $X_{\Sigma}$ is Fano if and only if $\deg(p) > 0$ for every $p \in PC(\Sigma)$.

$\blacklozenge$ $X_{\Sigma}$ is weak Fano if and only if $\deg(p) \geq 0 \ \forall \ p \in PC(\Sigma)$.

Ex.)

\begin{align*}
\omega_{P_2}(i) &= \omega_{P_2} \\
\omega_{P_1} &= \omega_{P_1} \\
\omega_{P_3} &= \omega_{P_3} \\
\omega_{P_4} &= \omega_{P_4} \\
\omega_{P_3} + \omega_{P_4} &= (\frac{1}{2}) + (\frac{1}{2}) = (1) \\
\omega_{P_1} + \omega_{P_2} &= (\frac{1}{2}) + (\frac{1}{2}) = (1) \\
\deg(P_1) &= 2 - 0 = 2 \\
\deg(P_2) &= \begin{cases} 
2 - a & (a \geq 0) \\
2 + a & (a < 0)
\end{cases} \\
\deg(P_2) > 0 \iff -1 < a < 1
\end{align*}
Conjecture [Cho-Lee-Masuda-P.J]

Let $X$ and $Y$ be smooth Fano toric varieties.
If there is a $c_1$-preserving cohomology ring isomorphism from $H^*(X; \mathbb{Z})$ to $H^*(Y; \mathbb{Z})$, then $X$ and $Y$ are isomorphic as toric varieties.

Ex.) The Hirzebruch surface $H_2 = \mathbb{P}(\mathcal{O} \oplus Y^2)$ is weak Fano.
There is a $c_1$-preserving cohomology ring isomorphism from $H^*(H_2)$ to $H^*(H_0)$. However, $H_0$ & $H_2$ are not isomorphic.

Ex.) There exist Fano Bott manifolds which are not isomorphic but diffeomorphic.

\[
\begin{bmatrix}
-1 \\
0 & -1 \\
1 & 1 & -1 \\
\end{bmatrix} \quad & \quad \begin{bmatrix}
-1 \\
0 & -1 \\
1 & -1 & -1 \\
\end{bmatrix}
\]
Fano generalized Bott manifolds.

\[ \begin{array}{c}
B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0, \\
P(\mathbb{C} \oplus \xi_m \oplus \cdots \oplus \xi^n_m) \xrightarrow{\mathbb{C}P^n} \{ \text{a point} \}
\end{array} \]

\[ \xi^i_k = \bigotimes_{\ell \leq j < i} \gamma_{\ell,j} \]

\[ \Rightarrow \{ a_{ij}^{k} \}_{1 \leq j < i \leq m} \text{ determines a generalized Bott manifold,} \]

and the primitive ray generators of \( B_m \) are the columns of the matrix

\[
\mathbf{a}_{\infty} = \begin{bmatrix} a_{1i}^k \\
\vdots \\
a_{mi}^k 
\end{bmatrix}
\]

\[
\Rightarrow e_{j}^{1} + \cdots + e_{j}^{n_{i}} + v_{j} = \sum_{i=j+1}^{m} \left( \lambda_{ij}^{0} v_{i} + \sum_{k=1}^{n_{i}} \lambda_{ij}^{k} e_{k}^{i} \right), \quad \prod_{k=0}^{n_{i}} \lambda_{ij}^{k} = 0, \lambda_{ij}^{k} \in \mathbb{Z}_{\geq 0}
\]

\[
B_m \text{ is Fano } \iff n_{j} - \sum_{i=j+1}^{m} \sum_{k=0}^{n_{i}} \lambda_{ij}^{k} > 0, \quad \forall j = 1, \ldots, m.
\]
Ex.)

\[
\begin{bmatrix}
1 & 1 & -1 & 0 \\
1 & 1 & a_1 & -1 \\
1 & a_2 & -1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
a_1 \\
a_2
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
a_1 \\
a_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
a_1 \\
a_2
\end{bmatrix}
\]

\[
= \lambda_1^0 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \lambda_1^1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_1^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[\lambda_1^0, \lambda_1^1, \lambda_1^2 \geq 0\]

\[\lambda_1^0 \lambda_1^1 \lambda_1^2 = 0\]

\[\lambda_1^0 + \lambda_1^1 + \lambda_1^2 < 3\]

If \(a_1, a_2\) are non-negative, then \(0 \leq a_1 + a_2 < 3\).

Otherwise, it is not easy to describe it neatly.

Fortunately, we may assume that \(a_{2,1}\) is nonnegative up to isomorphism.
Operations on generalized Bott matrices

\[ \begin{bmatrix} E_{m_1} & E_{m_2} & E_{m_3} & \cdots & E_{m_m} \end{bmatrix} \]

\[ \begin{bmatrix} a_{2,1} & -1 & \cdots & \cdots & -1 \\ a_{3,1} & a_{3,2} & \cdots & \cdots & -1 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & \cdots & a_{m_m} \end{bmatrix} \]

B(A): the generalized Bott manifold associated with A

We say that two generalized Bott matrices A and A' are isomorphic if B(A) and B(A') are isomorphic as toric varieties.

There are three operations on the set of generalized Bott matrices.

Op. 1 Fix \( i \in [m] \). Let F be the matrix obtained from E by changing one of the columns \( e_i^1, \ldots, e_i^k \) into \( w_i \). Then \( A' = FA \) is isomorphic to A.

Op. 2 Fix \( i \in [m] \). Let \( E_i^\pi \) be the matrix obtained from E by substituting \( e_i^1, \ldots, e_i^i \) with \( e_1^\pi, \ldots, e_i^\pi, e_i^{i+1} \). Then \( A' = E_i^\pi A \) is isomorphic to A.

Op. 3 Assume \( a_{k,k} = 0 \) for \( j < e < k < i \). Let \( E_{ij} \) be the matrix obtained from E by swapping \( e_i^1, \ldots, e_i^k \) and \( e_j^1, \ldots, e_j^k \). Then \( A' = E_{ij} A \) is isomorphic to A.
Let $A$ be a generalized Bott matrix having two columns.

If $\min \{ a_{2,k} \mid 1 \leq k \leq n \} = a_{2,1}$ is negative, then we let $F$ be the matrix obtained from $E$ by changing $e_{2,0}$ into $w_2$.

Then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + F^t A$ is a nonnegative matrix, i.e., $\begin{bmatrix} -1 & 0 \\ -1 & 0 \\ \vdots & \vdots \\ a_1 & -1 \\ \vdots & \vdots \\ a_n & -1 \end{bmatrix}$, $a_1, \ldots, a_n > 0$.

Using this fact, we can show the following.

**Theorem** For two-stage Fano generalized Bott manifolds $B$ & $B'$, if there is a $c_1$-preserving cohomology ring isomorphism from $H^\ast(B)$ to $H^\ast(B')$, then $B$ and $B'$ are isomorphic.

pf) Assume $B = B(n, (a_1, \ldots, a_m))$ and $B' = (n, (a'_1, \ldots, a'_m))$. Then

$\sigma_k(a_1, \ldots, a_m) = \sigma_k(a'_1, \ldots, a'_m)$ for $k = 1, \ldots, \min\{n, m\}$.

Since $\sigma_1(a_1, \ldots, a_m) < n$, we can show that $(a_1, \ldots, a_m) = (a'_1, \ldots, a'_m)$.  \[\Box\]
Recall [Choi–Suh–Masuda, 2010]

- Cohomological rigidity holds for two-stage generalized Bott manifolds.
- A cohomologically trivial generalized Bott manifold is diffeomorphic to a trivial generalized Bott manifold, $\prod \mathbb{C}P^{n_i}$.

**Theorem.** Let $B_m$ be a Fano generalized Bott manifold. If $H^*(B_m)$ is isomorphic to $H^*(\prod_{i=1}^{m} \mathbb{C}P^{n_i})$, then $B_m$ and $\prod_{i=1}^{m} \mathbb{C}P^{n_i}$ are isomorphic as toric varieties.

**pf.** Easy to check that it is true for $m=2$.

- The cohomology ring isomorphism $H^*(B_m) \to H^*(\prod_{i=1}^{m} \mathbb{C}P^{n_i})$
  
  induces the cohomology ring isomorphisms $H^*(B_{i_1, \ldots, i_{j-1}}) \to H^*(\prod_{i=1}^{m-1} \mathbb{C}P^{n_i})$

  and $H^*(B_{i_2, \ldots, i_{j}}) \to H^*(\prod_{i=1}^{m-1} \mathbb{C}P^{n_i})$. 

$\square$
Theorem  Let $B$ and $\tilde{B}$ be generalized Bott towers of fiber-dimension $(n_1, \ldots, n_m)$. Suppose that $B$ and $\tilde{B}$ are Fano and they satisfy one of the following:

Case 1: $n_1 > n_2 > \cdots > n_m > 1$ or

Case 2: the generalized Bott matrices $A$ and $\tilde{A}$ satisfy

$$\tilde{a}_{i,j}^1 \geq \tilde{a}_{i,j}^2 \geq \cdots \geq \tilde{a}_{i,j}^{n_i} \geq 0 \quad \text{and} \quad a_{i,j}^1 \geq a_{i,j}^2 \geq \cdots \geq a_{i,j}^{n_i} \geq 0$$

for all $1 \leq j < i \leq m$.

If there is a $c_1$-preserving graded ring isomorphism $\varphi$ from $H^*(\tilde{B})$ to $H^*(B)$ such that $[\varphi]$ is lower-triangular, then $B$ and $\tilde{B}$ are isomorphic.

Corollary  Assume $n_1 + \cdots + n_{i-1} \leq n_i$ for every $i = 2, \ldots, m$. If there is a $c_1$-preserving graded cohomology ring isomorphism between two Fano generalized Bott manifolds $B$ and $\tilde{B}$, then $B$ and $\tilde{B}$ are isomorphic.
Thank you!

ありがとうございます。