Cohomological rigidity on Fano generalized Bott manifolds

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Generalized Bott manifold

$$\begin{array}{ccccccccc} \mathcal{B}_m & \xrightarrow{\pi_m} & \mathcal{B}_{m-1} & \xrightarrow{\pi_{m-1}} & \cdots & \xrightarrow{\pi_2} & \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_0, \\ \| & & & & & & \| & & \| \\ P(\underline{\mathbb{C}} \oplus \xi_m^1 \oplus \cdots \oplus \xi_m^{n_m}) & & & & & & \mathbb{C}P^{n_1} & & \{\text{a point}\} \end{array}$$

$\mathcal{B}_m$: m-stage generalized Bott manifold

$\Rightarrow$ $\mathcal{B}_m$: a smooth projective toric variety of dim $N = \sum_{i=1}^{m} n_i$.

$\Rightarrow$ $\mathcal{B}_m / T^N \approx \Delta^{n_1} \times \Delta^{n_2} \times \cdots \times \Delta^{n_m}$

Ex). $n_i = 1 \ \forall i \Rightarrow \mathcal{B}_m$ is a Bott manifold.

- $\prod_{i=1}^{m} \mathbb{C}P^{n_i}$: trivial generalized Bott manifold.

Fano variety

In algebraic geometry, a smooth Fano variety is a smooth projective variety X whose anticanonical divisor $-K_X$ is ample.

i.e., the 1st Chern class can be represented by a positive definite differential form.

Ex.) • $\mathbb{C}P^n$ $\forall n$

• smooth hypersurfaces in $\mathbb{C}P^n$ of degree $\leq n$.

• Hirzebruch surfaces $\mathbb{P}(\underline{\mathbb{C}} \oplus \gamma^a)$, $-1 \leq a \leq 1$.

c_1-cohomological rigidity for Fano toric varieties

Conjecture [Cho-Lee-Masuda-P.]

Let X and Y be smooth Fano toric varieties.
If there is a c_1-preserving cohomology ring isomorphism from $H^*(X;\mathbb{Z})$ to $H^*(Y;\mathbb{Z})$, then X and Y are isomorphic as toric varieties.

Affirmative results

(1) Fano Bott manifolds (CLMP, arXiv: 2005.02740)

(2) $\dim_{\mathbb{C}} X \leq 4$ or Picard number $\geq 2\dim_{\mathbb{C}} X - 2$.
(Higashitani-Kurimoto-Masuda)

Today

① Characterization of smooth Fano toric varieties using the notion of primitive collections (Batyrev's work)

② Fano generalized Bott manifolds

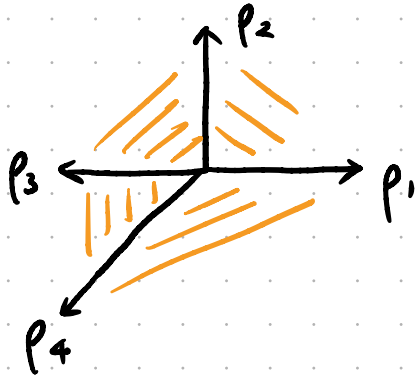
③ c_1 - cohomological rigidity for certain Fano generalized Bott manifolds.

Characterization of Fano toric varieties

Let Σ be a complete non-singular fan. the set of rays of Σ

A subset $P = \{\rho_1, \cdots, \rho_k\}$ of $\Sigma(1)$ is a primitive collection if P is not contained in $\sigma(1)$ for all $\sigma \in \Sigma$, but any proper subset is.

Ex.)



$\{\rho_1, \rho_3\}$ & $\{\rho_2, \rho_4\}$

Let u_ρ be the primitive integral vector generating the ray ρ.

For each primitive collection $P = \{\rho_1, \cdots, \rho_k\}$, $\sum_{i=1}^{k} u_{\rho_i}$ lies in the relative interior of a cone $\gamma \in \Sigma$.

$$\therefore \sum_{i=1}^{k} u_{\rho_i} = \sum_{\rho \in \gamma} a_\rho u_\rho \quad (a_\rho \in \mathbb{Z}_{\geq 0})$$

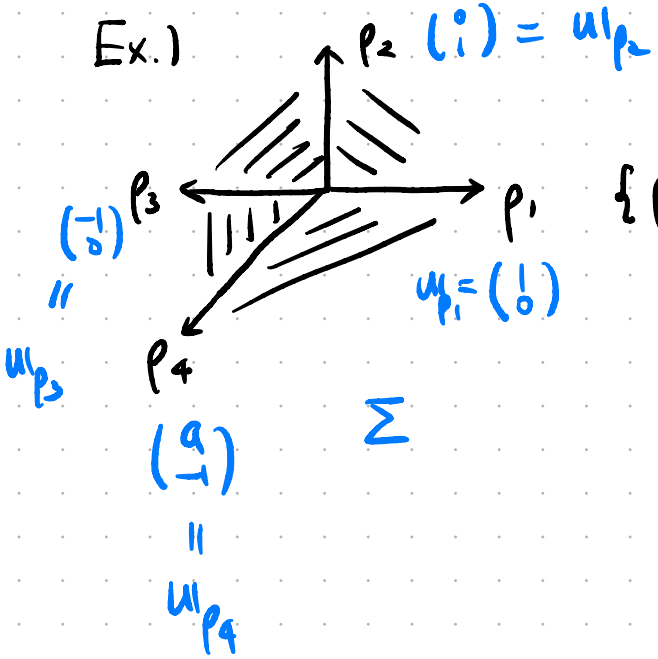
Define $\deg(P) = k - \sum_{\rho \in \gamma} a_\rho$.

Theorem [Batyrev]

Let X_Σ be the projective smooth toric variety.
Let $PC(\Sigma)$ be the set of primitive collections of Σ.
Then the toric variety X_Σ is Fano if and only if $\deg(P) > 0$ for every $P \in PC(\Sigma)$.

※ X_Σ is weak Fano if and only if $\deg(P) \geq 0$ $\forall P \in PC(\Sigma)$.

Ex.)



$\{\rho_1, \rho_3\}$ & $\{\rho_2, \rho_4\}$
P_1 P_2

$$u_{\rho_1} + u_{\rho_3} = \binom{1}{0} + \binom{-1}{0} = \binom{0}{0} \qquad \deg(P_1) = 2 - 0 = 2$$

$$u_{\rho_2} + u_{\rho_4} = \binom{0}{1} + \binom{a}{-1} = \binom{a}{0} = \begin{cases} a\binom{1}{0} & \text{if } a \geq 0 \\ -a\binom{-1}{0} & \text{if } a < 0 \end{cases}$$

$$\deg(P_2) = \begin{cases} 2 - a & (a \geq 0) \\ 2 + a & (a < 0) \end{cases} \qquad \deg(P_2) > 0 \Leftrightarrow -1 \leq a \leq 1$$

Conjecture [Cho-Lee-Masuda-P.]

Let X and Y be smooth Fano toric varieties.
If there is a c_1-preserving cohomology ring isomorphism from $H^*(X;\mathbb{Z})$ to $H^*(Y;\mathbb{Z})$, then X and Y are isomorphic as toric varieties.

Ex.) The Hirzebruch surface $\mathcal{H}_2 = P(\mathbb{C} \oplus \gamma^2)$ is weak Fano.

There is a c_1-preserving cohomology ring isomorphism from $H^*(\mathcal{H}_2)$ to $H^*(\mathcal{H}_0)$. However, $\mathcal{H}_0$ & $\mathcal{H}_2$ are not isomorphic.

Ex.) There exist Fano Bott manifolds which are not isomorphic but diffeomorphic.

$$\begin{bmatrix} -1 & & \\ 0 & -1 & \\ 1 & 1 & -1 \end{bmatrix} \quad \& \quad \begin{bmatrix} -1 & & \\ 0 & -1 & \\ 1 & -1 & -1 \end{bmatrix}$$

Fano generalized Bott manifolds.

$$\begin{array}{ccccccccc} \mathcal{B}_m & \xrightarrow{\pi_m} & \mathcal{B}_{m-1} & \xrightarrow{\pi_{m-1}} & \cdots & \xrightarrow{\pi_2} & \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_0, \\ \| & & & & & & \| & & \| \\ P(\underline{\mathbb{C}} \oplus \xi_m^1 \oplus \cdots \oplus \xi_m^{n_m}) & & & & & & \mathbb{C}P^{n_1} & & \{\text{a point}\} \end{array}$$

$$\begin{array}{ccccccc} \gamma_{i,j} & & & & \gamma_{j+1,j} & & \\ \| & & & & \| & & \\ \pi_i^* \cdots \pi_{j+1}^* \pi_j^* \gamma_{j,j} & & & & \pi_{j+1}^* \gamma_{j,j} & & \gamma_{j,j} \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathcal{B}_i & \xrightarrow{\pi_i} & \cdots & \to & \mathcal{B}_{j+1} & \xrightarrow{\pi_{j+1}} & \mathcal{B}_j \end{array}$$

$$\xi_k^i = \bigotimes_{1 \le j < i} \gamma_{i-1,j}^{\otimes a_{ij}^k}$$

$$\xi_k^i \downarrow \mathcal{B}_{i-1}$$

$\Rightarrow \{a_{ij}^k\}_{\substack{1 \le j < i \le m \\ 1 \le k \le n_i}}$ determines a generalized Bott manifold,

and the primitive ray generators of $\mathcal{B}_m$ are the columns of the matrix

Char. matrix

$$\begin{array}{c} \left[\begin{array}{ccccc|ccccc} E_{n_1} & & & & & -\mathbb{1} & & & & \\ & E_{n_2} & & & & a_{2,1} & -\mathbb{1} & & & \\ & & E_{n_3} & & & a_{3,1} & a_{3,2} & -\mathbb{1} & & \\ & & & \ddots & & \vdots & \vdots & \vdots & \ddots & \\ & & & & E_{n_m} & a_{m,1} & a_{m,2} & a_{m,3} & \cdots & -\mathbb{1} \end{array}\right] \\ e_1^1, \cdots, e_1^{n_1}, e_2^1, \cdots, e_{m-1}^{n_{m-1}}, e_m^1, \cdots, e_m^{n_m}, w_1, w_2, \cdots, w_{m-1}, w_m \end{array}$$

$$a_{ij} = \begin{bmatrix} a_{ij}^1 \\ \vdots \\ a_{ij}^{n_i} \end{bmatrix}$$

$$\Rightarrow e_j^1 + \cdots + e_j^{n_j} + w_j = \sum_{i=j+1}^{m} \left(\lambda_{ij}^0 w_i + \sum_{k=1}^{n_i} \lambda_{ij}^k e_i^k \right), \quad \prod_{k=0}^{n_i} \lambda_{ij}^k = 0, \quad \lambda_{ij}^k \in \mathbb{Z}_{\geq 0}$$

$$\boxed{\mathcal{B}_m \text{ is Fano} \iff n_j - \sum_{i=j+1}^{m} \sum_{k=0}^{n_i} \lambda_{ij}^k > 0 \quad \forall j = 1, \cdots, m.}$$

Ex.)

$$\left[\begin{array}{cc|cc|c|c} 1 & & & & -1 & 0 \\ & 1 & & & -1 & 0 \\ \hline & & 1 & & a_1 & -1 \\ & & & 1 & a_2 & -1 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$= \lambda^0_{1,1} \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} + \lambda^1_{1,1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \lambda^2_{1,1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda^0_{1,1},\ \lambda^1_{1,1},\ \lambda^2_{1,1} \geq 0$$

$$\lambda^0_{1,1}\ \lambda^1_{1,1}\ \lambda^2_{1,1} = 0$$

$$\lambda^0_{1,1} + \lambda^1_{1,1} + \lambda^2_{1,1} < 3$$

If a_1 & a_2 are non-negative, then $0 \leq a_1 + a_2 < 3$.

Otherwise, it is not easy to describe it neatly.

Fortunately, we may assume that $a_{1,1}$ is nonnegative up to isomorphism.

Operations on generalized Bott matrices

$$\underbrace{\left[\begin{array}{ccccc} E_{n_1} & & & & \\ & E_{n_2} & & & \\ & & E_{n_3} & & \\ & & & \ddots & \\ & & & & E_{n_m} \end{array}\right.}_{=:\,E}\ \boxed{\begin{array}{ccccc} -1 & & & & \\ a_{2,1} & -1 & & & \\ a_{3,1} & a_{3,2} & -1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & -1 \end{array}}\Bigg]$$

$=:$ A (generalized Bott matrix)

$\mathcal{B}(A)$: the generalized Bott manifold associated with A

We say that two generalized Bott matrices A and A' are isomorphic if $\mathcal{B}(A)$ and $\mathcal{B}(A')$ are isomorphic as toric varieties.

There are three operations on the set of generalized Bott matrices.

<u>Op. 1</u> Fix $i \in [m]$. Let F be the matrix obtained from E by changing one of the columns $e_i^1, \cdots, e_i^{n_i}$ into w_i. Then $A' = F^{-1}A$ is isomorphic to A.

<u>Op. 2</u> Fix $i \in [m]$. Let E_i^{π} be the matrix obtained from E by substituting $e_i^1, \cdots, e_i^{n_i}$ with $e_i^{\pi(1)}, \cdots, e_i^{\pi(n_i)}$. Then $A' = E_i^{\pi} A$ is isomorphic to A.

<u>Op. 3</u> Assume $a_{k,l} = 0$ for $j \leq l < k \leq i$. Let $E_{i,j}$ be the matrix obtained from E by swapping $e_i^1, \cdots, e_i^{n_i}$ and $e_j^1, \cdots, e_j^{n_j}$. Then $A' = E_{i,j} A$ is isomorphic to A.

Let A be a generalized Bott matrix having two columns.

If $\min\{a_{2,1}^k \mid 1 \le k \le n_2\} = a_{2,1}^{k_0}$ is negative, then we let F be the matrix obtained from E by changing $e_2^{k_0}$ into v_2.

Then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + F^{-1}A$ is a nonnegative matrix. i.e., $\begin{bmatrix} -1 & 0 \\ \vdots & \vdots \\ -1 & 0 \\ a_1 & -1 \\ \vdots & \vdots \\ a_n & -1 \end{bmatrix}$, $a_1, \cdots, a_n \geq 0$.

Using this fact, we can show the following.

<u>Theorem</u> For two-stage Fano generalized Bott manifolds $\mathcal{B}$ & $\mathcal{B}'$, if there is a c_1-preserving cohomology ring isomorphism from $H^*(\mathcal{B})$ to $H^*(\mathcal{B}')$, then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic.

pf) Assume $\mathcal{B} = \mathcal{B}(n, (a_1, \cdots, a_m))$ and $\mathcal{B}' = (n, (a_1', \cdots, a_m'))$. Then

$\sigma_k(a_1, \cdots, a_m) = \sigma_k(a_1', \cdots, a_m')$ for $k = 1, \cdots, \min\{n, m\}$.

Since $\sigma_1(a_1, \cdots, a_m) < n$, we can show that $(a_1, \cdots, a_m) = (a'_{\pi(1)}, \cdots, a'_{\pi(m)})$. $\blacksquare$

Recall [Choi–Suh–Masuda, 2010]

- Cohomological rigidity holds for two-stage generalized Bott manifolds
- A cohomologically trivial generalized Bott manifold is diffeomorphic to a trivial generalized Bott manifold, $\prod \mathbb{C}P^{n_i}$.

Theorem Let B_m be a Fano generalized Bott manifold. If $H^*(B_m)$ is isomorphic to $H^*(\prod_{i=1}^{m} \mathbb{C}P^{n_i})$, then B_m and $\prod_{i=1}^{m} \mathbb{C}P^{n_i}$ are isomorphic as toric varieties.

pf) • Easy to check that it is true for $m=2$.

- The cohomology ring isomorphism $H^*(B_m) \longrightarrow H^*(\prod_{i=1}^{m} \mathbb{C}P^{n_i})$ induces the cohomology ring isomorphisms $H^*(B_{\{1,\cdots,m-1\}}) \to H^*(\prod_{i=1}^{m-1} \mathbb{C}P^{n_i})$ and $H^*(B_{\{2,\cdots,m\}}) \to H^*(\prod_{i=1}^{m-1} \mathbb{C}P^{n_i})$. ∎

Theorem *Let $\mathcal{B}_\bullet$ and $\widetilde{\mathcal{B}}_\bullet$ be generalized Bott towers of fiber-dimension $(n_1, \ldots, n_m)$. Suppose that $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are Fano and they satisfy one of the following:*

Case 1: $n_1 > n_2 > \cdots > n_m > 1$ *or*

Case 2: *the generalized Bott matrices A and $\widetilde{A}$ satisfy*

$$\widetilde{a}^1_{i,j} \geq \widetilde{a}^2_{i,j} \geq \cdots \geq \widetilde{a}^{n_i}_{i,j} \geq 0 \quad \text{and} \quad a^1_{i,j} \geq a^2_{i,j} \geq \cdots \geq a^{n_i}_{i,j} \geq 0$$

for all $1 \leq j < i \leq m$.

If there is a c_1-preserving graded ring isomorphism φ from $H^*(\widetilde{\mathcal{B}})$ to $H^*(\mathcal{B})$ such that $[\varphi]$ is lower-triangular, then $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are isomorphic.

Corollary *Assume $n_1 + \cdots + n_{i-1} \leq n_i$ for every $i = 2, \ldots, m$. If there is a c_1-preserving graded cohomology ring isomorphism between two Fano generalized Bott manifolds $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, then $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are isomorphic.*

Thank you!

ありがとうございます。