# Generalized equivariant cohomology theory of weighted Grassmans

**Soumen Sarkar**
Indian Institute of Technology Madras
*soumen@iitm.ac.in*
(Joint work with Koushik Brahma)

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and $GL(d, \mathbb{C})$ the set of all non singular complex matrix of order $d$.

For two matrix $A, B \in M_d(n, d, \mathbb{C})$,

$$A \sim B \text{ if and only if } A = BT$$

for some $T \in GL(d, \mathbb{C})$.

# Grassmanns

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The space $\mathrm{Gr}(d,n)$ is a $d(n-d)$-dimensional smooth manifold.

# CW structures on Grassmanns

## Definition

*A* **Schubert symbol** $\lambda$ *for* $d < n$ *is a sequence of* $d$ *integers* $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ *such that* $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$.

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The length of $\lambda$ is defined by

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Consider $\mathbb{C}^k = \{(z_1, \ldots, z_k, 0, \ldots, 0) \in \mathbb{C}^n\}$.

The Schubert cell $E(\lambda)$ is defined by

$$E(\lambda) := \{X \in \mathrm{Gr}(d,n) \ : \ \dim(X \cap \mathbb{C}^{\lambda_j}) = j, \dim(X \cap \mathbb{C}^{\lambda_j - 1}) = j-1;$$

$$\forall\, j = 1, 2, \ldots, n\}.$$

# CW structures Grassmanns

## Proposition ( Milnor and Stasheff, (Chapter-6) [1])

$$E(\lambda) \cong \begin{bmatrix} * & * & \dots & * \\ * & * & \dots & * \\ 1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & * \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix}$$

# CW structures Grassmanns

**Proposition ( Milnor and Stasheff, (Chapter-6) [2])**

*Let $\lambda^0, \ldots, \lambda^m$ be the Schubert symbols for $d < n$. Then*

$$Gr(d,n) = E(\lambda^0) \sqcup E(\lambda^1) \sqcup \cdots \sqcup E(\lambda^m).$$

[2] Milnor, John W.; Stasheff, James D. Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.

# Weighted Grassmanns

Denote an element $A \in M_d(n, d; \mathbb{C})$ as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

where $\mathbf{a}_i \in \mathbb{C}^d$ for $i = 1, \ldots, n$.

# Weighted Grassmanns (Definition)

For $W := (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$, we define an equivalence relation $\sim_w$ on $M_d(n, d; \mathbb{C})$ by

$$\begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{pmatrix} \sim_w \begin{pmatrix} t^{w_1}\mathbf{a_1} \\ t^{w_2}\mathbf{a_2} \\ \vdots \\ t^{w_n}\mathbf{a_n} \end{pmatrix} T$$

for $T \in GL(d, \mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T)$.

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for $T \in GL(d, \mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T)$.

We denote the identification space by

$$\mathrm{WGr}(d, n) := \frac{\mathrm{M}_d(n, d; \mathbb{C})}{\sim_w}.$$

# Weighted Grassmanns

The natural $(\mathbb{C}^*)^n$ action on $\mathbb{C}^n$ induces a $(\mathbb{C}^*)^n$ action on $\mathrm{WGr}(d, n)$.

[3] Abe, Hiraku; Matsumura, Tomoo. *Equivariant cohomology of weighted Grassmannians and weighted Schubert classes*. Int. Math. Res. Not. IMRN 2015, no. 9, 2499-2524.

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## Proposition

*The space $WGr(d,n)$ has an orbifold structure where the charts can be chosen $(\mathbb{C}^*)^n$-invariant.*

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## Remark

*Our definition is equivalent to the definition of weighted Grassman by Abe and Matsumora* [3]

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# $q$-CW structure on weighted Grassmanns

For $W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$, let

$$c_i := a + \sum_{j=1}^{d} w_{\lambda_j^i} \tag{2}$$

where $\{\lambda^i = (\lambda_1^i, \lambda_2^i, \ldots, \lambda_d^i) : i = 0, \ldots, m = \binom{n}{d} - 1\}$ are Schubert symbols.

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## Proposition (Abe and Matsumora [4])

*There is a q-cell structure on $WGr(d, n)$ for $0 < d < n$ as follows*

$$WGr(d, n) = \frac{E(\lambda^0)}{G(c_0)} \sqcup \frac{E(\lambda^1)}{G(c_1)} \sqcup \cdots \sqcup \frac{E(\lambda^m)}{G(c_m)}.$$

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# Order on Schubert symbols

**Definition**

*Let $\alpha$ and $\lambda$ be Schubert symbols for $d < n$. We say that $\alpha < \lambda$ if $\ell(\alpha) < \ell(\lambda)$ (called Bruhat order), otherwise we use the dictionary order if $\ell(\alpha) = \ell(\lambda)$.*

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This gives a total order on the set of all Schubert symbols.

# Divisive weighted Grassmanns

For $m = \binom{n}{d} - 1$, let

$$\lambda^0 < \lambda^1 < \lambda^2 < \cdots < \lambda^m \tag{3}$$

be a total order on the Schubert symbols for $d < n$.

[4] M. Harada, T. S. Holm, N. Ray, and G. Williams, *The equivariant K-theory and cobordism rings of divisive weighted projective spaces*, Tohoku Math. J. (2) 68:4 (2016), 487-513.

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## Definition

*A weighted Grassmann orbifold $WGr(d, n)$ is called divisive if $c_i$ divides $c_{i-1}$ for all $i = 1, 2, \ldots, m$ where $c_i$ is defined in (2).*

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All divisive weighted projective spaces are divisive weighted Grassmans $WGr(1, n)$.

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## Example

Consider the Grassmann $\mathrm{WGr}(2,4)$ for $w = (6,1,1,1)$ and $a = 3$.

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We have the ordering on the 6 Schubert symbols given by

$$\lambda^0 = 12 < \lambda^1 = 13 < \lambda^2 = 14 < \lambda^3 = 23 < \lambda^4 = 24 < \lambda^5 = 34.$$

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Then

$$c_0 = a + w_1 + w_2 = 10, c_2 = 10, c_3 = 10, c_4 = 5, c_5 = 5, c_6 = 5.$$

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Then $c_i$ divides $c_{i-1}$ for all $i = 1, 2, \ldots, 6$.

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**Theorem (Brahma-S)**

*If* $\mathrm{WGr}(d, n)$ *is a divisive weighted Grassmann then it has an invariant CW-structure with only even dimensional cells.*

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**Corollary**

*If* $\mathrm{WGr}(d,n)$ *is divisive then* $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ *has no torsion and concentrated in even degrees.*

# Weighted Grassmanns

If $\mathrm{WGr}(d,n)$ is divisive then each $\frac{E(\lambda^i)}{G(c_i)} \cong E(\lambda^m)$ with respect to $T^n$-action. If $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a Schubert symbol, then denote

$$X_\lambda := x_{\lambda_1} + \cdots + x_{\lambda_d} \in \mathbb{Z}[x_1, \ldots, x_n].$$

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## Proposition (Brahma-S)

*Let $\lambda^i$ and $\lambda^j$ be two Schubert symbols with $i < j$ . If $WGr(d,n)$ be a divisive Grassmann, then $d_{ij} := \frac{c_i}{c_j} \in \mathbb{Z}$ and the corresponding axial function ($T^n$ representation) is $(X_{\lambda^i} - d_{ij} X_{\lambda^j})$.*

# Cohomology ring of weighted Grassmanns

## Theorem (Brahma-S)

*The integral equivariant cohomology of a divisive weighted Grassmann $WGr(d,n)$ corresponding to the order $\lambda^0 < \cdots < \lambda^m$ is given by*

$$H_{T^n}(WGr(d,n),\mathbb{Z})$$
$$= \{(f_i) \in \oplus_{i=0}^{m}\mathbb{Z}[x_1,x_2,\ldots x_n] : (X_{\lambda^i} - d_{ij}X_{\lambda^j})|(f_j - f_i)$$
$$\text{whenever } |\lambda^j \cap \lambda^i| = d-1 \text{ and } i \leq j\}.$$

**Arigatogozaimashita**

**Dhanyabad**

**Gamsahabnida**

**ThankYou**