Lower bound for Buchstaber invariants of real universal complexes

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Outline

Definition (Moment-angle complex)

Given a simplicial complex K on $[m] = \{1, \ldots, m\}$, the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ and the *moment-angle complex* $\mathcal{Z}_K$ associated to K are defined as:

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{I \subset K} (D^1, S^0)^I \subseteq (D^1)^m \qquad \mathcal{Z}_K = \bigcup_{I \subset K} (D^2, S^1)^I \subseteq (D^2)^m$$

where $(X, A)^I = \{(x_1, \ldots, x_m) \in X^m, x_i \in A \text{ if } i \notin I\}$.

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Naturally, $\mathbb{R}\mathcal{Z}_K$ admits a $\mathbb{Z}_2^m$-action by coordinate-wise sign permutation and $\mathcal{Z}_K$ admits a T^m-action by coordinate-wise rotation. However, these actions are NOT free unless K is the empty complex.

Definition (Buchstaber invariant)

For $\mathbb{R}\mathcal{Z}_K$ and $\mathcal{Z}_K$ associated to a simplicial complex K on $[m]$:
(1) The *real Buchstaber invariant* $s_{\mathbb{R}}(K)$ is the maximal rank of a subgroup $H \subseteq \mathbb{Z}_2^m$ s.t. the restricted action $H \curvearrowright \mathbb{R}\mathcal{Z}_K$ is free;
(2) The *Buchstaber invariant* $s(K)$ is the maximal rank of a toric subgroup $G \subseteq T^m$ s.t. the restricted action $G \curvearrowright \mathcal{Z}_K$ is free.

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- Existence of free group action $\Longleftrightarrow$ Certain symmetry.
- $S_{\mathbb{R}}(K)$ and $S(K)$ measure the degree of symmetry of $\mathbb{R}\mathcal{Z}_K$ and $\mathcal{Z}_K$ respectively.

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- (Fukukawa-Masuda 2009)
  Computation of $s_{\mathbb{R}}(K)$ where K is the skeleta of a simplex.

For arbitrary $(n-1)$-dimensional simplicial complex K with m vertices, there exists general bounds for $s_{\mathbb{R}}(K)$ and $s(K)$:

$$m - \gamma(K) \leq s(K) \leq s_{\mathbb{R}}(K) \leq m - n \leq log_2(\sum_i \beta^i)$$

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- A special case of Halperin-Carlsson conjecture proved by Cao-Lü and Ustinovskiy independently.

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- Related with the existence of small cover and quasitoric manifold over a simple polytope P when $K = (\partial P)^*$.

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- Involutions on T^m and $\mathcal{Z}_K$ induced by complex conjugation have fixed point sets $\mathbb{Z}_2^m$ and $\mathbb{R}\mathcal{Z}_K$ respectively.

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- (Ayzenberg 2009) Systematic viewpoint: relations among *generalized chromatic numbers*.

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(2) Denote Δ_i^∞ as the complex consisting of countably many vertices and all simplices with less or equal to $(i+1)$ vertices. Then *dimension* corresponds to a non-degenerate simplicial map $\nu : K \to \Delta^\infty_{\dim(K)}$.

Definition (Universal complex)

The *real universal complex* $\mathcal{K}_1^n$ (resp. *universal complex* $\mathcal{K}_2^n$) is defined on the set of primitive vectors in $\mathbb{Z}_2^n$ (resp. $\mathbb{Z}^n$) s.t. $[\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}]$ is a simplex $\Longleftrightarrow \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\}$ is part of a basis.

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- Both $\mathcal{K}_1^n$ and $\mathcal{K}_2^n$ are $(n-1)$-dimensional.
- A $\mathbb{Z}_2^n$-coloring (resp. $\mathbb{Z}^n$-coloring) on a simpicial complex K is equivalent to a non-degenerate simplicial map from K to $\mathcal{K}_1^n$ (resp. $\mathcal{K}_2^n$).

Notation

(1) $r_{\mathbb{R}}(K) = \min\{r \mid \exists\ \mathbb{Z}_2^r\text{-coloring on } K\}$;
(2) $r(K) = \min\{r \mid \exists\ \mathbb{Z}^r\text{-coloring on } K\}$;
(3) $\Delta(K) = r(K) - r_{\mathbb{R}}(K)$.

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- $\Delta(K) = r(K) - r_{\mathbb{R}}(K) = s_{\mathbb{R}}(K) - s(K)$ can be regarded as an obstruction to the *Lifting Problem*.
- $s(\mathcal{K}_1^n) = 2^n - 1 - n - \Delta(\mathcal{K}_1^n) = 2^n - 1 - r(\mathcal{K}_1^n)$. Thus, lower bound estimation of $s(\mathcal{K}_1^n)$ is equivalent to upper bound estimation of $\Delta(\mathcal{K}_1^n)$ or $r(\mathcal{K}_1^n)$.

By restriction and composition of certain colorings:

Proposition (*Sun 2017*)

(1) Monotonicity: $\Delta(\mathcal{K}_1^n) \leq \Delta(\mathcal{K}_1^{n+1})$.

(2) Universal bound: $r_{\mathbb{R}}(K) \leq r_{\mathbb{R}}(\mathcal{K}_1^n) = n \Longrightarrow \Delta(K) \leq \Delta(\mathcal{K}_1^n)$.

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Upper bound estimation of $\Delta(\mathcal{K}_1^n)$:

- Construction of $\lambda : \mathcal{K}_1^n \to \mathcal{K}_2^N$;
- Determinant calculation in both $\mathbb{Z}_2$ and $\mathbb{Z}$.

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Example ($GL(4, \mathbb{Z}_2)$ elements with integral determinant ± 3)

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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- $GL(n, \mathbb{Z}_2)/\sim$ by permutations of rows and columns;
- The non-degenerate simplicial map $\lambda : \mathcal{K}_1^4 \to \mathcal{K}_2^5$ satisfies:
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Example (Illustration of $\lambda : \mathcal{K}_1^4 \to \mathcal{K}_2^5$)

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \mapsto \quad \lambda(M_1) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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- $vt(\mathcal{K}_1^5) = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4 \sqcup V_5$ s.t. V_i consists of primitive vectors with $(5-i)$ zeros;
- The non-degenerate simplicial map $\lambda : \mathcal{K}_1^5 \to \mathcal{K}_2^7$ satisfies:
  (1) $p_j \circ \lambda = id_j$ for $j = 1, 2, 3, 4, 5$;

  $$(2)\ p_6 \circ \lambda(\boldsymbol{v_i}) = \phi(\boldsymbol{v_i}) = \begin{cases} 0 & \boldsymbol{v_i} \in V_1 \\ 1 & \boldsymbol{v_i} \in V_2 \sqcup V_3 \sqcup V_4 \\ 2 & \boldsymbol{v_i} \in V_5 \end{cases}$$

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- The existence and classification problem of solutions to only one equation is not hard.
- The number of equations grow rapidly as n increases.
- $|GL(n, \mathbb{Z}_2)| = \prod_{i=0}^{n-1}(2^n - 2^i)$.

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- A special case of *Hadamard maximum determinant problem*: calculate the maximal integral determinant of a square matrix with elements restricted in a given set $S \subseteq \mathbb{Z}$.
- This bound may NOT be sharp as the actual upper bounds for n =4, 5, 6 are 3, 5 and 9 respectively.
- (Zivkovic 2006) Computational results in dimension no more than 10 with the aid of computer.

Lemma 1

Any $M \in Mat(4, \mathbb{Z}_2)$ with integral determinant ± 3 is equivalent to one of the following matrices:

(1) $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ *(2)* $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

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Fact 1

For all $M \in GL(5, \mathbb{Z}_2)$ with integral determinant ± 5, there exists a (4×4) minor M_{ij} with element $m_{ij} = 1$ s.t. $\det M_{ij} = \pm 3$.

Lemma 2

Any $M \in Mat(4,\mathbb{Z}_2)$ with integral determinant ± 2 belongs to one of the following types up to equivalence:

(1) $\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & y_1 & y_2 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

where $x_1, x_2, x_3 \in \{0,1\}$ and $y_1, y_2 \in \{0,1\}$.

Fact 2

For all $M \in GL(5, \mathbb{Z}_2)$ with integral determinant ± 3, there exists a (4×4) minor M_{ij} with element $m_{ij} = 1$ s.t. $\det M_{ij} = \pm 3$ or ± 2.

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In total, there are 3 equivalent classes of $GL(5,\mathbb{Z}_2)$ elements with integral determinant ± 5 and 51 equivalent classes of $GL(5,\mathbb{Z}_2)$ elements with integral determinant ± 3.

Example

$$X_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- With additional restrictions, ϕ is NOT enough.
- The value of ψ on $V_1 \sqcup V_5$ is irrelevant.
- The value of ψ can be taken modulo 3.

Theorem (Erokhovets 2009)

For a simplicial complex K on $[m]$, if $\{\omega_i\}_{i=1}^{l}$ is a collection of minimal non-simplices s.t. $\bigcup_{i=1}^{l} vt(\omega_i) = [m]$, then $r(K) \leq \sum_{i=1}^{l} \mathrm{dim}\omega_i$.

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Take $K = \mathcal{K}_1^n (n \geq 2)$ and choose two arbitrary primitive vectors $\boldsymbol{x}, \boldsymbol{y} \in vt(\mathcal{K}_1^n)$, then theorem above can be applied to the case of $\omega_1 = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\}$ and $\omega_i = \{\boldsymbol{a}_i, \boldsymbol{a}_i+\boldsymbol{x}, \boldsymbol{a}_i+\boldsymbol{y}, \boldsymbol{a}_i+\boldsymbol{x}+\boldsymbol{y}\}$ with $\boldsymbol{a}_i \in vt(\mathcal{K}_1^n)$.

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Take $K = \mathcal{K}_1^n (n \geq 2)$ and choose two arbitrary primitive vectors $\boldsymbol{x}, \boldsymbol{y} \in vt(\mathcal{K}_1^n)$, then theorem above can be applied to the case of $\omega_1 = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\}$ and $\omega_i = \{\boldsymbol{a}_i, \boldsymbol{a}_i+\boldsymbol{x}, \boldsymbol{a}_i+\boldsymbol{y}, \boldsymbol{a}_i+\boldsymbol{x}+\boldsymbol{y}\}$ with $\boldsymbol{a}_i \in vt(\mathcal{K}_1^n)$.

Corollary

For $n \geq 2$, $\Delta(\mathcal{K}_1^n) \leq 3 \cdot 2^{n-2} - 1 - n$, i.e., $s(\mathcal{K}_1^n) \geq 2^{n-2}$.

Theorem B (S. 2021)

For $n \geq 2$, $\Delta(\mathcal{K}_1^n) \leq 2^{n-2} + 1 - n$, i.e., $s(\mathcal{K}_1^n) \geq 3 \cdot 2^{n-2} - 2$.

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For a fixed partition $\{\omega_i\}_{i=1}^{2^{n-2}}$, a non-degenerate simplicial map $\lambda : \mathcal{K}_1^n \to \mathcal{K}_2^{2^{n-2}+1}$ can be induced by the vertex map:

$$x \mapsto e_1 \qquad y \mapsto e_2 \qquad a_i \mapsto e_{i+2}$$
$$x + y \mapsto e_1 + e_2$$
$$a_i + x \mapsto e_1 + e_{i+2}$$
$$a_i + y \mapsto e_2 + e_{i+2}$$
$$a_i + x + y \mapsto e_1 + e_2 + e_{i+2}$$

where $\{e_j\}_{j=1}^{2^{n-2}+1}$ is the standard basis.

Example ($n = 4$)

Take primitive vectors $\boldsymbol{x}, \boldsymbol{y}$ as $(1,0,0,0)^{\mathrm{T}}$ and $(0,1,0,0)^{\mathrm{T}}$ respectively, then the vertex map is defined as follow:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Theorem B (S. 2021)

For $n \geq 2$, $\Delta(\mathcal{K}_1^n) \leq 2^{n-2} + 1 - n$, i.e., $s(\mathcal{K}_1^n) \geq 3 \cdot 2^{n-2} - 2$.

- This estimation is sharp for $n = 2, 3, 4$.

Theorem B (S. 2021)

For $n \geq 2$, $\Delta(\mathcal{K}_1^n) \leq 2^{n-2} + 1 - n$, i.e., $s(\mathcal{K}_1^n) \geq 3 \cdot 2^{n-2} - 2$.

- This estimation is sharp for $n = 2, 3, 4$.
- Similar construction can NOT give out better results.

Example (Start from choosing three primitive vectors)

$R = \begin{pmatrix} R_3 & * \\ 0 & I_{2^{n-3}-1} \end{pmatrix}$ where R_3 may equal to $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ due to necessary column subtractions.

- What is the exact value of $\Delta(\mathcal{K}_1^5)$?
  computer aid v.s. search for contradiction

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- How to further estimate $\Delta(\mathcal{K}_1^n)$?
exponential or linear or bounded