Homotopy Classification of 4-dimensional Toric Orbifolds

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Motivation and Question

Cohomological rigidity problem

For two (quasi)toric manifolds M and N,

$$H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z}) \overset{?}{\Longrightarrow} M \cong N.$$

- No counter-example has beed found so far.

Question

For two toric orbifolds X and Y,

$$H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z}) \overset{?}{\Longrightarrow} X \cong Y.$$

- $\exists$ many counter-examples in weighted projective spaces.
- e.g., $H^*(\mathbb{CP}^2_{1,2,3};\mathbb{Z}) \cong H^*(\mathbb{CP}^2_{1,1,6};\mathbb{Z})$, but $\mathbb{CP}^2_{1,2,3} \ncong \mathbb{CP}^2_{1,1,6}$. [1]
- However, $\mathbb{CP}^2_{1,2,3} \simeq \mathbb{CP}^2_{1,1,6}$.[1]

[1] A. Bahri, M. Franz, D. Notbohm, and N. Ray, *The classification of weighted projective spaces.* Fund. Math. 220 (2013), no. 3, 217–226.

Question and Theorem

Modified question

For two toric orbifolds X and Y,

$$H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z}) \overset{(?)}{\Longrightarrow} X \simeq Y$$

- This is true for all weighted projective spaces.

Theorem (Fu–So–S)

Let X and Y be two 4-dimensional toric orbifolds such that $H^3(X;\mathbb{Z})$ and $H^3(Y;\mathbb{Z})$ have no 2-torsion. Then,

$$X \simeq Y \text{ if and only if } H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z}).$$

Toric orbifolds

- P: simple polytope.
- $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^{\dim P}$ such that

  $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$ is linearly independent whenever $\bigcap_{j=1}^{k} F_{i_j} \neq \emptyset$.

Definition (Constructive definition)

$$X(P, \lambda) = (P \times T^{\dim P})/_{\sim_\lambda},$$

where $(x, t) \sim_\lambda (y, s)$, whenever $x = y$ and $t^{-1}s \in T_{F(x)}$.

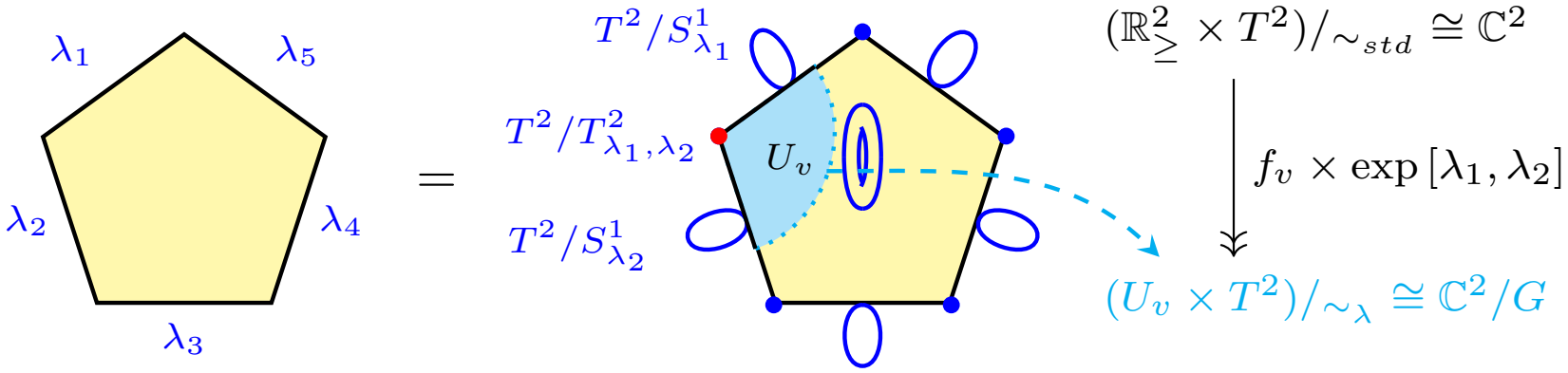
When $\dim P = 2$,

- P: $(n+2)$-gon. $(n \geq 1)$
- $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^2$ such that

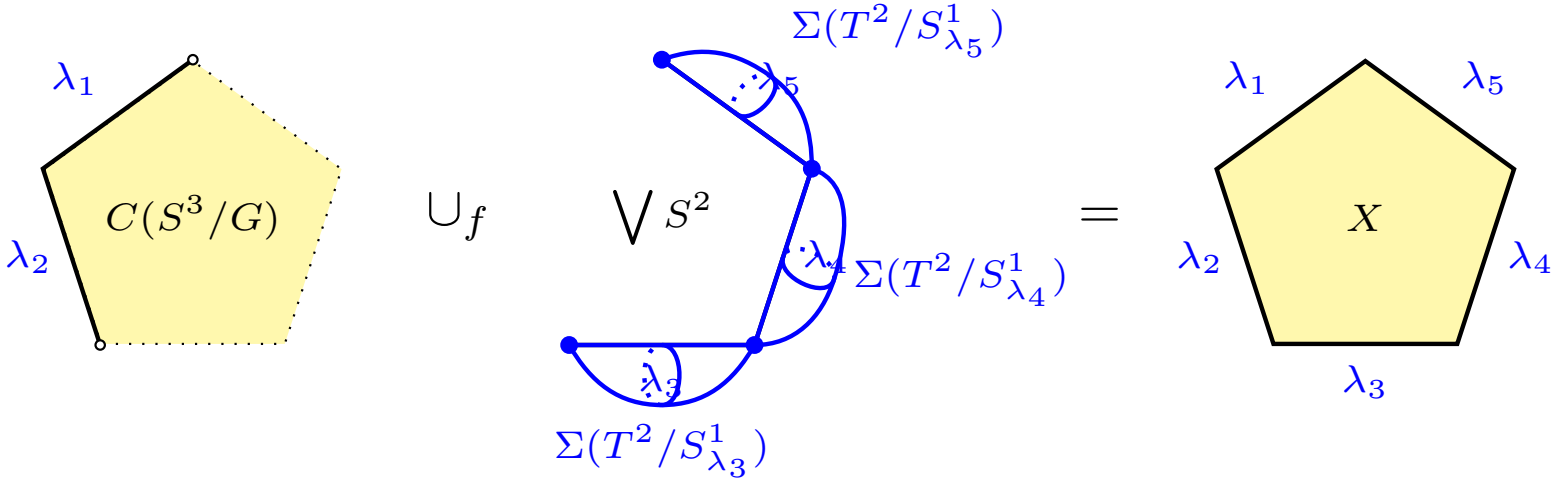
  $\{\lambda(F_i), \lambda(F_{i+1})\}$ is linearly independent for $i = 1, \ldots, n+2$.

Pictorial description of a 4-dimensional toric orbifold

1. T^2-orbits and the orbifold structure.



2. Cofibration. $(S^3/G \xrightarrow{f} \bigvee S^2 \longrightarrow X)$



Topology of 4-dimensional toric orbifold X.

1. $\pi_1(X) = 1$.
2. homology groups are:

i	0	1	2	3	4
$H_i(X;\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}^n \oplus \mathbb{Z}_m$	0	$\mathbb{Z}$

   where $(m = \text{gcd}\,\{\det\,[\lambda(F_i)\ \lambda(F_j)] : 1 \leq i < j \leq n+2\})$.

3. (Homology decomposition theorem implies...)

$$X \simeq \left(\bigvee_n S^2 \vee P^3(m) \right) \cup_f e^4$$

   for some attaching map $f\colon S^3 \to \bigvee_n S^2 \vee P^3(m)$,
   where $P^3(m)$ denotes the mod m Moore space, namely, the cofiber of the degree m map on S^2.

Cellular basis

Let $\mathscr{C}_{n,m}$ be the category of mapping cones

$$C_f := \left(\bigvee_{i=1}^{n} S_i^2 \vee P^3(m) \right) \cup_f e^4.$$

	$i = 2$	$i = 3$	$i = 4$
$H_i(C_f;\mathbb{Z})$	$\mathbb{Z}^n \langle \mu_1, \ldots, \mu_n \rangle \oplus \mathbb{Z}_m \langle \nu \rangle$ • $\mu_i \leftrightarrow S_i^2$; • $\nu \leftrightarrow$ the 2-cell of $P^3(m)$	0	$\mathbb{Z} \langle \epsilon \rangle$
$H^i(C_f;\mathbb{Z})$	$\mathbb{Z}^n \langle u_1, \ldots, u_n \rangle$ • u_i : dual to μ_i.	$\mathbb{Z}_m \langle v \rangle$	$\mathbb{Z} \langle e \rangle$
$H^i(C_f;\mathbb{Z}_m)$	$(\mathbb{Z}_m)^{n+1} \langle \bar{u}_1, \ldots, \bar{u}_n, \bar{v} \rangle$ • $\bar{u}_i$: mod m image of u_i; • $\bar{v}$: dual to ν.	$\mathbb{Z}_m \langle \beta(\bar{v}) \rangle$	$\mathbb{Z}_m \langle \bar{e} \rangle$

(1) Cup product in $H^*(C_f;\mathbb{Z})$ and $H^*(C_f;\mathbb{Z}_m)$

Definition ('cellular cup product representation' of C_f)

$$M_{cup}(f) := (A, \mathbf{b}, c) \in \mathsf{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m$$

such that

1. $u_i \cup u_j = a_{ij}e$, where $A = (a_{ij})_{1\le i,j\le n}$;
2. $\bar{u}_i \cup \bar{v} = b_i\bar{e}$, where $\mathbf{b} = (b_1, \ldots, b_n)$;
3. $\bar{v} \cup \bar{v} = c\bar{e}$.

(2) Cellular map $\psi\colon C_{f'} \to C_f$ for $C_f, C_{f'} \in \mathscr{C}_{n,m}$

Definition ('cellular map representation' of $\psi\colon C_{f'} \to C_f$)

$$M(\psi) := (W, \mathbf{y}, z) \in \mathsf{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m$$

such that

1. $\psi^*_{\mathbb{Z}}(u_j) = \sum_{i=1}^n w_{ij}u'_i$, where $W = (w_{ij})_{1\le i,j\le n}$;
2. $\psi^*_{\mathbb{Z}_m}(\bar{v}) = \sum_{i=1}^n y_i\bar{u}'_i + z\bar{v}'$, where $\mathbf{y} = (y_1, \ldots, y_n)$, $z \in \mathbb{Z}_m$.

Homotopy theory of complexes in $\mathscr{C}_{n,1}$

- We consider a mapping cone C_f with $f\colon S^3 \to \bigvee_{i=1}^n S_i^2$.
- $\mathscr{C}_{n,1}$ includes
  - all quasitoric manifolds of dimension 4;
  - in general, 4-dimensional toric orbifolds $X(P,\lambda)$ with $H^3 = 0$, e.g. (P,λ) contains at least one 'smooth' vertex.
  - other CW-complexes which may not be realizable as toric spaces.
- $M_{cup}(C_f) \in \mathsf{Mat}_n(\mathbb{Z})$.
- $M(\psi) \in \mathsf{Mat}_n(\mathbb{Z})$ for a cellular map $\psi\colon C_f \to C_{f'}$.

- 'Hilton–Milnor theorem' implies

$$f \simeq \sum_{i=1}^{n} a_i \eta_i + \sum_{1 \le j < k \le n} a_{jk} \omega_{jk},$$

for some integers a_i and a_{jk}, where

- $\eta_i \colon S^3 \xrightarrow{\text{Hopf}} S^2_i \hookrightarrow \bigvee_{\ell=1}^{n} S^2_\ell$
- $\omega_{jk} \colon S^3 \xrightarrow{[\iota_1, \iota_2]} S^2_j \vee S^2_k \hookrightarrow \bigvee_{\ell=1}^{n} S^2_\ell$

<u>Lemma</u>. $M_{cup}(C_f) = \begin{pmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ a_{12} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_n. \end{pmatrix}$

Theorem

$C_f \simeq C_{f'}$ if and only if $H^*(C_f) \cong H^*(C_{f'})$.

Homotopy theory of complexes in $\mathscr{C}_{n,m}$ with $m > 1$

- We consider a mapping cone C_f with $f\colon S^3 \to \bigvee_{i=1}^n S_i^2 \vee P^3(m)$.
- $\mathscr{C}_{n,m}$ includes all toric orbifolds of dimension 4.
- 'Hilton–Milnor theorem' implies

$$f \simeq \sum_{i=1}^n a_i \eta_i + \sum_{1 \le j < k \le n} a_{jk}\omega_{jk} + \eta_P + \sum_{i=1}^n \omega_{iP},$$

for some integers a_i and a_{jk}, where

- $\eta_i\colon S^3 \xrightarrow{\text{Hopf}} S_i^2 \hookrightarrow \bigvee_{\ell=1}^n S_\ell^2$,
- $\omega_{jk}\colon S^3 \xrightarrow{[\iota_1,\iota_2]} S_j^2 \vee S_k^2 \hookrightarrow \bigvee_{\ell=1}^n S_\ell^2$,
- $\eta_P\colon S^3 \xrightarrow{g} P^3(m) \hookrightarrow \bigvee_{\ell=1}^n S_\ell^2 \vee P^3(m)$,
- $\omega_{iP}\colon S^3 \xrightarrow{g_i} P^4(m) \xrightarrow{[\kappa_1,\kappa_2]} S_i^2 \vee P^3(m) \hookrightarrow \bigvee_{\ell=1}^n S_\ell^2 \vee P^3(m)$.

Result for $\mathscr{C}_{n,m}$

Proposition

For $C_f \in \mathscr{C}_{n,m}$, let

$$M_{cup}(C_f) = ((a_{ij})_{1\le i,j\le n}, (b_1, \dots, b_n), c) \in \mathit{Mat}_n(\mathbb{Z}) \oplus (\mathbb{Z}_m)^n \oplus \mathbb{Z}_m.$$

Then, $C_f \simeq \hat{C} \vee P^3(m)$ for some $\hat{C} \in \mathscr{C}_{n,1}$ if and only if the system of mod-m linear equations

$$(\star) \qquad \begin{cases} a_{11}y_1 + \cdots + a_{1n}y_n & \equiv & -b_1 \\ \qquad\qquad \vdots & & \\ a_{n1}y_1 + \cdots + a_{nn}y_n & \equiv & -b_n \\ b_1y_1 + \cdots + b_ny_n & \equiv & -c \end{cases} \pmod{m}$$

has a solution.

Main result (1) for 4-dimensional toric orbifolds

Theorem 1

Let X be 4-dimensional toric orbifold such that $H^3(X;\mathbb{Z}) = \mathbb{Z}_m$ for some odd $m > 1$. Then,

$$X \simeq \hat{X} \vee P^3(m)$$

for some $\hat{X} \in \mathscr{C}_{n,1}$ such that $H^i(X;\mathbb{Z}) = H^i(\hat{X};\mathbb{Z})$ for $i \neq 3$.

Sketch of proof.

1. $m = p_1^{r_1} \cdots p_s^{r_s}$ prime factorization with odd primes $p_1, \ldots, p_s$.
2. $X \simeq_{(p_i)} \hat{X} \vee P^3(p_i^{r_i})$ for each $i = 1, \ldots, s$.
3. mod $p_i^{r_i}$ version of $(\star)$ has a solution for each i.
4. Chinese Remainder thm gives a solution of mod m version of $(\star)$.
5. The result follows from the Proposition above. $\square$

Main result (2) for 4-dimensional toric orbifolds

Theorem 2

Let X and Y be two 4-dimensional toric orbifolds such that $H^3(X;\mathbb{Z})$ and $H^3(Y;\mathbb{Z})$ have no 2-torsion. Then,

$$X \simeq Y \text{ if and only if } H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z}).$$

Sketch of proof.

1. Since 'only if' part is obvious, we show 'if' part.
2. Theorem 1 implies
$$X \simeq \hat{X} \vee P^3(m) \quad \text{and} \quad Y \simeq \hat{Y} \vee P^3(m)$$
for some $\hat{X}, \hat{Y} \in \mathscr{C}_{n,1}$.
3. Hypothesis implies $H^*(\hat{X}) \cong H^*(\hat{Y})$.
4. The result for $\mathscr{C}_{n,1}$ shows $\hat{X} \simeq \hat{Y}$, which yields $X \simeq Y$. $\square$

Thank you for your attention.