Poincaré series of the spaces of commuting elements in Lie groups

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March 26, 2021

Space of homomorphisms

- Let G be a compact connected Lie group with rank n.

Definition For a discrete group π, let

$$\mathrm{Hom}(\pi, G)$$

be the space of homomorphisms from π to G.

Connection of geometry

- $\mathrm{Hom}(\pi, G)/G$ is called character variety.
- $\mathrm{Hom}(\pi, G)$ is identified with the based moduli space of flat G-bundles over a space X with $\pi_1(X) \cong \pi$.

Space of commuting elements

We work with $\pi = \mathbb{Z}^m$.

There is a natural homeomorphism

$$\mathrm{Hom}(\mathbb{Z}^m, G) \cong \{(g_1, \ldots, g_m) \in G^m | g_i g_j = g_j g_i \text{ for all } i, j\}.$$

So we call $\mathrm{Hom}(\mathbb{Z}^m, G)$ space of commuting elements.

Previous work

There are previous work on the topology of $\mathrm{Hom}(\mathbb{Z}^m, G)$:

- Cohomology
  - Rational cohomology in terms of an invariant ring by Baird '07
  - Formula for Poincaré series by Ramras and Stafa '19
- Stable decomposition
  - Stable splitting by Adem, Bendersky, Cohen and Gitler '10
  - Rigorous calculation of homology of $\mathrm{Hom}(\mathbb{Z}^m, SU(2))$ by Crabb '08 and Baird, Jeffrey, and Selick '11
- Homotopy group
  - Fundamental group by Gomez, Pettet and Souto '12
  - Second homotopy group by Adem, Gomez and Gritschacher '20
- ... and so on.

- Let T be a maximal torus of G.
- Let W be the Weyl group of G.
- Let $\mathbb{F}$ be a field of characteristic zero or prime to the order of W.

Theorem(Baird) There is an isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F}) \cong (H^*(G/T; \mathbb{F}) \otimes H^*(T; \mathbb{F})^{\otimes m})^W,$$

where $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ denote the path-component of $\mathrm{Hom}(\mathbb{Z}^m, G)$ containing $(1, \ldots, 1) \in G^m$.

Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G)$

Let $d_1 \le d_2 \le \cdots \le d_n$ be the characteristic degrees of G i.e.

$$H^*(BT;\mathbb{F})^W \cong \mathbb{F}[x_{2d_1}, x_{2d_2}, \ldots, x_{2d_n}]$$

where $|x_i| = i$.

Theorem(Ramras-Stafa) The Poincaré series of $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ is given by

$$P(\mathrm{Hom}(\mathbb{Z}^m, G)_1; t) = \frac{1}{|W|}\prod_{i=1}^{r}(1 - t^{2d_i})\sum_{w\in W}\frac{\det(1_n + tw)^m}{\det(1_n - t^2 w)},$$

where the determinant is computed by the canonical representation of W on the Lie algebra of T.

Example

$$P(\mathrm{Hom}(\mathbb{Z}^2, SU(2))_1; t) = 1 + t^2 + 2t^3$$

$$P(\mathrm{Hom}(\mathbb{Z}^2, SU(3))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + t^6 + 2t^7 + 3t^8$$

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^2, SU(4))_1; t) &= 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 6t^8 + 6t^9 \\ &\quad + 8t^{10} + 6t^{11} + 7t^{12} + 2t^{13} + 3t^{14} + 4t^{15} \end{aligned}$$

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^2, SU(5))_1; t) &= 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 10t^8 \\ &\quad + 14t^9 + 13t^{10} + 16t^{11} + 22t^{12} + 18t^{13} + 21t^{14} \\ &\quad + 20t^{15} + 22t^{16} + 18t^{17} + 14t^{18} + 14t^{19} + 10t^{20} \\ &\quad + 10t^{21} + 3t^{22} + 4t^{23} + 5t^{24}. \end{aligned}$$

Partition

Definition A sequence of integers

$$\lambda = (\lambda_1, \dots \lambda_l)$$

is a partition of a positive integer k, denoted by $\lambda \vdash k$, if λ satisfies $1 \leq \lambda_1 \leq \cdots \leq \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = k$.

For an integer partition

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \dots, \underbrace{\lambda_l, \dots, \lambda_l}_{n_l}),$$

we define an integer $\theta(\lambda)$ as

$$\theta(\lambda) \coloneqq \lambda_1 \cdots \lambda_l n_1! \cdots n_l!.$$

Results

For a positive integer k, let

$$q_k(t) = \frac{2}{1 + (-1)^k t^k}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k \leq n$, let

$$q_\lambda(t) = q_{\lambda_1}(t) \cdots q_{\lambda_l}(t).$$

Theorem(Kishimoto-T)
The Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, SU(n))_1$ is given by

$$P(\mathrm{Hom}(\mathbb{Z}^2, SU(n))_1; t) = \frac{1-t}{1+t} \prod_{i=2}^{n} (1 - t^{2i}) \sum_{k=n-1}^{n} \sum_{\lambda \vdash k} \frac{(-1)^{n+k}}{\theta(\lambda)} q_\lambda(t).$$

General case of $SU(n)$

For a positive integer k, let

$$q_k^m(t) = (-1)^{m(k-1)} t^{(m-2)k} + \frac{(1 + (-1)^{k+1} t^k)^m}{1 - t^{2k}}.$$

For $\lambda = (\lambda_1, \dots, \lambda_l) \vdash k \leq n$, let

$$q_\lambda^{m,n}(t) = t^{(m-2)(n-k)} q_{\lambda_1}^m(t) \cdots q_{\lambda_l}^m(t)$$

where we set $q_\lambda^{m,n}(t) = t^{(m-2)n}$ if λ is the empty partition.

Theorem The Poincaré series of $\mathrm{Hom}(\mathbb{Z}^m, SU(n))_1$ is given by

$$P(\mathrm{Hom}(\mathbb{Z}^m, SU(n))_1; t) =$$

$$\begin{cases} \displaystyle\prod_{i=1}^n (1 - t^{2i}) \sum_{k=n-1}^n \sum_{\lambda \vdash k} \frac{(-1)^{n+k}}{\theta(\lambda)(1+t)^m} q_\lambda^{m,n}(t) & (m \text{ even}) \\ \displaystyle\prod_{i=1}^n (1 - t^{2i}) \sum_{k=0}^n \sum_{\lambda \vdash k} \frac{(-1)^k}{\theta(\lambda)(1+t)^m} q_\lambda^{m,n}(t) & (m \text{ odd}). \end{cases}$$

Top term

- For other classical groups G, we obtain similar formulae for $P(\mathrm{Hom}(\mathbb{Z}^m, G)_1)$.
- For exceptional Lie groups G, we calculate $P(\mathrm{Hom}(\mathbb{Z}^2, G)_1)$ with an assistance of computer.

$\Rightarrow$ By combining the calculations, we obtain the following theorem.

Theorem(Kishimoto-T)
Let G be a compact connected Lie group with simple factors $G_1, G_2, \ldots G_k$. Then the top term of the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^2, G)_1$ is

$$(\mathrm{rank}\, G_1 + 1) \cdots (\mathrm{rank}\, G_k + 1) t^{\dim G + \mathrm{rank}\, \pi_1(G)}.$$

Rational hyperbolicity of $\mathrm{Hom}(\mathbb{Z}^m, G)$

Definition Let X be a simply-connected space. X is called rationally hyperbolic when $\sum_{k \leq n} \dim(\pi_k(X) \otimes \mathbb{Q})$ is of exponential growth, and called rationally elliptic when $\sum_n \dim(\pi_n(X) \otimes \mathbb{Q})$ is finite.

Theorem Every simply-connected finite complex X is either rationally hyperbolic or elliptic.

Theorem If a simply-connected space X is rationally elliptic, then $H^*(X; \mathbb{Q})$ satisfies the Poincaré duality.

Theorem (Kishimoto-T)

When G is simply-connected, $\mathrm{Hom}(\mathbb{Z}^m, G)_1$ *is rationally hyperbolic for* $m \geq 2$.

Further problems and yet more results on cohomology

Further problems

- Is there a topological interpretation of the identity for the top term of $P(\mathrm{Hom}(\mathbb{Z}^2, G)_1; t)$?
- Can we describe torsion in $H^*(\mathrm{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Z})$?

We have more results about $H^*(\mathrm{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F})$.

- Minimal generating set.
- Homology stability in the best possible range.
- Complete determination in the rank two case.