Poincaré series of the spaces of commuting elements in Lie groups

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March 26, 2021
Let $G$ be a compact connected Lie group with rank $n$.

**Definition** For a discrete group $\pi$, let

$$\text{Hom}(\pi, G)$$

be the space of homomorphisms from $\pi$ to $G$.

**Connection of geometry**

- $\text{Hom}(\pi, G)/G$ is called character variety.
- $\text{Hom}(\pi, G)$ is identified with the based moduli space of flat $G$-bundles over a space $X$ with $\pi_1(X) \cong \pi$. 
We work with \( \pi = \mathbb{Z}^m \).

There is a natural homeomorphism

\[
\text{Hom}(\mathbb{Z}^m, G) \cong \{(g_1, \ldots, g_m) \in G^m | g_ig_j = g_jg_i \text{ for all } i, j \}\.
\]

So we call \( \text{Hom}(\mathbb{Z}^m, G) \) space of commuting elements.
Previous work

There are previous work on the topology of $\text{Hom}(\mathbb{Z}^m, G)$:

- **Cohomology**
  - Rational cohomology in terms of an invariant ring by Baird ‘07
  - Formula for Poincaré series by Ramras and Stafa ‘19

- **Stable decomposition**
  - Stable splitting by Adem, Bendersky, Cohen and Gitler ‘10
  - Rigorous calculation of homology of $\text{Hom}(\mathbb{Z}^m, SU(2))$ by Crabb ‘08 and Baird, Jeffrey, and Selick ‘11

- **Homotopy group**
  - Fundamental group by Gomez, Pettet and Souto ‘12
  - Second homotopy group by Adem, Gomez and Gritschacher ‘20

- ... and so on.
Cohomology of $\text{Hom}(\mathbb{Z}^m, G)$

- Let $T$ be a maximal torus of $G$.
- Let $W$ be the Weyl group of $G$.
- Let $F$ be a field of characteristic zero or prime to the order of $W$.

**Theorem** (Baird) There is an isomorphism

$$H^*(\text{Hom}(\mathbb{Z}^m, G)_1; F) \cong (H^*(G/T; F) \otimes H^*(T; F)^{\otimes m})^W,$$

where $\text{Hom}(\mathbb{Z}^m, G)_1$ denote the path-component of $\text{Hom}(\mathbb{Z}^m, G)$ containing $(1, \ldots, 1) \in G^m$. 
Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the characteristic degrees of $G$ i.e.

$$H^*(BT; \mathbb{F})^W \cong \mathbb{F}[x_{2d_1}, x_{2d_2}, \ldots, x_{2d_n}]$$

where $|x_i| = i$.

**Theorem** (Ramras-Stafa) The Poincaré series of $\text{Hom}(\mathbb{Z}^m, G)_1$ is given by

$$P(\text{Hom}(\mathbb{Z}^m, G)_1; t) = \frac{1}{|W|} \prod_{i=1}^r (1 - t^{2d_i}) \sum_{w \in W} \frac{\det(1_n + tw)^m}{\det(1_n - t^2w)},$$

where the determinant is computed by the canonical representation of $W$ on the Lie algebra of $T$. 

Example

\[ P(\text{Hom}(\mathbb{Z}^2, SU(2))_1; t) = 1 + t^2 + 2t^3 \]

\[ P(\text{Hom}(\mathbb{Z}^2, SU(3))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + t^6 + 2t^7 + 3t^8 \]

\[ P(\text{Hom}(\mathbb{Z}^2, SU(4))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 6t^8 + 6t^9 + 8t^{10} + 6t^{11} + 7t^{12} + 2t^{13} + 3t^{14} + 4t^{15} \]

\[ P(\text{Hom}(\mathbb{Z}^2, SU(5))_1; t) = 1 + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 8t^7 + 10t^8 + 14t^9 + 13t^{10} + 16t^{11} + 22t^{12} + 18t^{13} + 21t^{14} + 20t^{15} + 22t^{16} + 18t^{17} + 14t^{18} + 14t^{19} + 10t^{20} + 10t^{21} + 3t^{22} + 4t^{23} + 5t^{24} . \]
Definition A sequence of integers

$$\lambda = (\lambda_1, \ldots, \lambda_l)$$

is a partition of a positive integer $k$, denoted by $\lambda \vdash k$, if $\lambda$ satisfies $1 \leq \lambda_1 \leq \cdots \leq \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = k$.

For an integer partition

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1}, \ldots, \underbrace{\lambda_l, \ldots, \lambda_l}_{n_l})$$

we define an integer $\theta(\lambda)$ as

$$\theta(\lambda) := \lambda_1 \cdots \lambda_l n_1! \cdots n_l!.$$
Results

For a positive integer $k$, let

$$q_k(t) = \frac{2}{1 + (-1)^k t^k}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k \leq n$, let

$$q_\lambda(t) = q_{\lambda_1}(t) \cdots q_{\lambda_l}(t).$$

**Theorem** (Kishimoto-T)

The Poincaré series of $\text{Hom}(\mathbb{Z}^2, SU(n))_1$ is given by

$$P(\text{Hom}(\mathbb{Z}^2, SU(n))_1; t) = \frac{1 - t}{1 + t} \prod_{i=2}^{n} (1 - t^{2i}) \sum_{k=n-1}^{n} \sum_{\lambda \vdash k} \frac{(-1)^{n+k}}{\theta(\lambda)} q_\lambda(t).$$
General case of $SU(n)$

For a positive integer $k$, let

$$q_k^m(t) = (-1)^{m(k-1)} t^{(m-2)k} + \frac{(1 + (-1)^{k+1}t^k)^m}{1 - t^{2k}}.$$ 

For $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k \leq n$, let

$$q_{\lambda}^{m,n}(t) = t^{(m-2)(n-k)} q_{\lambda_1}^m(t) \cdots q_{\lambda_l}^m(t)$$

where we set $q_{\lambda}^{m,n}(t) = t^{(m-2)n}$ if $\lambda$ is the empty partition.

**Theorem** The Poincaré series of $\text{Hom}(\mathbb{Z}^m, SU(n))_1$ is given by

$$P(\text{Hom}(\mathbb{Z}^m, SU(n))_1; t) =$$

$$\begin{cases} 
\prod_{i=1}^n (1 - t^{2i}) \sum_{k=n-1}^{n} \sum_{\lambda \vdash k} (-1)^{n+k} \frac{1}{\theta(\lambda)(1 + t)^m} q_{\lambda}^{m,n}(t) & (m \text{ even}) \\
\prod_{i=1}^n (1 - t^{2i}) \sum_{k=0}^{n} \sum_{\lambda \vdash k} (-1)^k \frac{1}{\theta(\lambda)(1 + t)^m} q_{\lambda}^{m,n}(t) & (m \text{ odd}).
\end{cases}$$
For other classical groups $G$, we obtain similar formulae for $P(\text{Hom}(\mathbb{Z}^m, G)_1)$.

For exceptional Lie groups $G$, we calculate $P(\text{Hom}(\mathbb{Z}^2, G)_1)$ with an assistance of computer.

By combining the calculations, we obtain the following theorem.

**Theorem (Kishimoto-T)**

Let $G$ be a compact connected Lie group with simple factors $G_1, G_2, \ldots G_k$. Then the top term of the Poincaré series of $\text{Hom}(\mathbb{Z}^2, G)_1$ is

$$(\text{rank } G_1 + 1) \cdots (\text{rank } G_k + 1)t^{\dim G + \text{rank } \pi_1(G)}.$$
Rational hyperbolicity of $\text{Hom}(\mathbb{Z}^m, G)$

**Definition** Let $X$ be a simply-connected space. $X$ is called rationally hyperbolic when $\sum_{k \leq n} \dim(\pi_k(X) \otimes \mathbb{Q})$ is of exponential growth, and called rationally elliptic when $\sum_{n} \dim(\pi_n(X) \otimes \mathbb{Q})$ is finite.

**Theorem** Every simply-connected finite complex $X$ is either rationally hyperbolic or elliptic.

**Theorem** If a simply-connected space $X$ is rationally elliptic, then $H^*(X; \mathbb{Q})$ satisfies the Poincaré duality.

**Theorem (Kishimoto-T)**

\textit{When $G$ is simply-connected, $\text{Hom}(\mathbb{Z}^m, G)_1$ is rationally hyperbolic for $m \geq 2$.}
Further problems and yet more results on cohomology

Further problems

- Is there a topological interpretation of the identity for the top term of $P(\text{Hom}(\mathbb{Z}^2, G)_1; t)$?
- Can we describe torsion in $H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{Z})$?

We have more results about $H^*(\text{Hom}(\mathbb{Z}^m, G)_1; \mathbb{F})$.

- Minimal generating set.
- Homology stability in the best possible range.
- Complete determination in the rank two case.