Equivariant cohomology ring GKM graph modeled by $T^n \times S^1$-action on $T^*\mathbb{C}^n$ (Based on joint work with Shintaro Kuroki)

V. Uma

I.I.T Chennai

Conference on Toric Topology
2021
Osaka

Oriented Graph with legs

- $\Gamma = (\mathcal{V}, \mathcal{E})$ $\quad \mathcal{V}$–set of vertices ; $\mathcal{E} = E \cup L$ where E–set of edges, L–set of legs
- $\forall e \in E$ let $i(e)$–initial vertex , $t(e)$–terminal vertex
- $\overline{e}$-oppositely directed edge with $i(\overline{e}) = t(e)$ and $t(\overline{e}) = i(e)$.
- A leg $l \in L$ has only initial vertex and no terminal vertex.
- For $p \in \mathcal{V}$ let
$$\mathcal{E}_p := \{f \in \mathcal{E} \mid i(f) = p\}$$
- We say Γ-regular m-valent if $|\mathcal{E}_p| = m$ for every $p \in \mathcal{V}$.

Motivation

- (Γ, α)-GKM graphs and its graph equivariant cohomology $H(\Gamma, \alpha)$ were defined by Guillemin and Zara ([GZ]) to enable combinatorial study of the topology of GKM manifolds. In particular,

$$H_T^*(M) \simeq H(\Gamma, \alpha)$$

- Torus graphs were defined by Maeda, Masuda and Panov ([MMP]) to study the cohomology ring structure of torus manifolds.

GKM graph with legs

- $\Gamma = (\mathcal{V}, \mathcal{E})$- regular m-valent graph
- $\nabla = \{\nabla_e \mid e \in E\}$ called *connection* if $\nabla_e : \mathcal{E}_{i(e)} \to \mathcal{E}_{t(e)}$ is a bijection satisfying:
-
  - $\nabla_{\bar{e}} = \nabla_e^{-1}$;
  - $\nabla_e(e) = \bar{e}$.
- $\alpha : \mathcal{E} \longrightarrow H^2(BT) = Hom(T^n, S^1)$-called *axial function* satisfying:
  - $\alpha(\bar{e}) = \pm\alpha(e) \ \forall \ \ e \in E$;
  - $\alpha(\mathcal{E}_p) = \{\alpha(\epsilon) \mid \epsilon \in \mathcal{E}_p\}$– *pairwise linearly independent* $\forall \ \ p \in \mathcal{V}$,
  - $\alpha(\epsilon) - \alpha(\nabla_e(\epsilon)) \equiv 0 (\mathrm{mod}\ \alpha(e))$ *congruence relation* $\forall \ \ e \in E$ and $\forall \ \ \epsilon \in \mathcal{E}_{i(e)}$.
- Then $\mathcal{G} = (\Gamma, \nabla, \alpha)$ is called a *GKM graph with legs*.

Motivation from theory of toric hyperKahler varieties

- By the theory of *toric HyperKahler varieties* ([BD00], [K00],[HP]) the tangential representation on each fixed point set is isomorphic to T^n-action on $T^*(\mathbb{C}^n)$ given by $(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, w_1, \ldots, w_n) :=$ $(t_1 \cdot z_1, \ldots, t_n \cdot z_n, t_1^{-1} \cdot w_1, \ldots, t_n^{-1} \cdot w_n)$, where $(t_1, \ldots, t_n) \in T^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $(w_1, \ldots, w_n) \in T_z^*(\mathbb{C}^n)$.
- Harada and Proudfoot [HP] found a residual S^1 action on the toric hyperKahler variety so that the tangential representation at each fixed point may be regarded as $T^*(\mathbb{C}^n)$ with $T^n \times S^1$-action given by $(t_1, \ldots, t_n, r) \cdot (z_1, \ldots, z_n, w_1, \ldots, w_n) :=$ $(t_1 \cdot z_1, \ldots, t_n \cdot z_n, r \cdot t_1^{-1} \cdot w_1, \ldots, r \cdot t_n^{-1} \cdot w_n)$, where $r \in S^1$.

Relation with GKM graphs with legs

- Thus the toric hyperKahler varieties with $T^n \times S^1$-action give rise to 2n-valent GKM graphs with legs with axial functions

$$\{\alpha_1, \ldots, \alpha_n, -\alpha_1 + x, \ldots, -\alpha_n + x\}$$

with $\langle \alpha_1, \ldots, \alpha_n \rangle \simeq H^2(BT^n)$ and $\langle x \rangle \simeq H^2(BS^1)$.

GKM graphs *modelled by $T^n \times S^1$-action on $T^*(\mathbb{C}^n)$*

- $\mathcal{G} = (\Gamma, \alpha, \nabla)$–be a $2n$-valent GKM graph with legs with axial function
- $\alpha : \mathcal{E} \longrightarrow H^2(BT^n \times BS^1) \simeq \mathfrak{t}^*_{\mathbb{Z}} \oplus \mathbb{Z} \cdot x$ satisfying the following two conditions:
  1. $\forall \ p \in \mathcal{V}$
  $$\mathcal{E}_p = \{\epsilon_1^+, \ldots, \epsilon_n^+, \epsilon_1^-, \ldots, \epsilon_n^-\}$$
  such that $\alpha(\epsilon_j^+) + \alpha(\epsilon_j^-) = x$ for $1 \leq j \leq n$.
  2. $\{\alpha(\epsilon_1^+), \ldots, \alpha(\epsilon_n^+), x\}$ span $\mathfrak{t}^*_{\mathbb{Z}} \oplus \mathbb{Z} \cdot x$.
- The pair $\{\epsilon_j^+, \epsilon_j^-\}$ such that $\alpha(\epsilon_j^+) + \alpha(\epsilon_j^-) = x$ is called *a 1-dimensional pair* in $\mathcal{E}_p$ and x is called a *residual basis*.

Hyperplanes in $\mathcal{G}$

- Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ be a $T^*(\mathbb{C}^n)$-modelled GKM graph.
- A $(2n-2)$-valent GKM subgraph $\mathbb{L} := (L, \alpha^L, \nabla^L)$ is called a *hyperplane* if $\mathbb{L}$ is

    1. $T^*(\mathbb{C}^{n-1})$-modelled with residual basis x.
    2. $\mathbb{L}$ is maximal with these properties.

Example

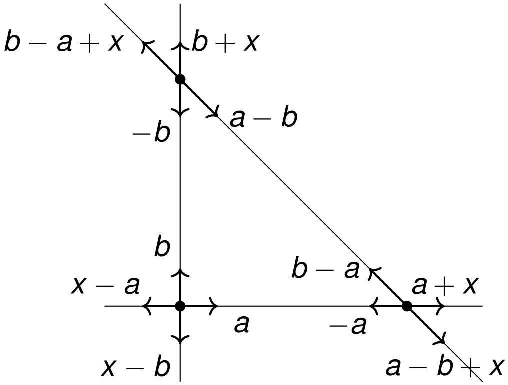
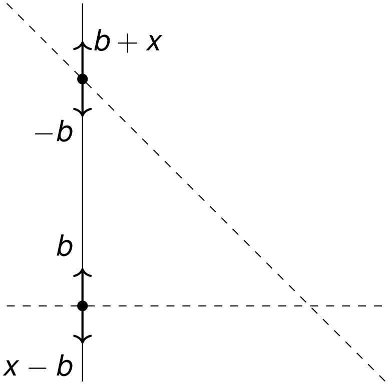


Figure: Example of GKM graph with legs associated to $T^*(\mathbb{CP}^n)$

Example of hyperplane



Graph Equivariant Cohomology

- Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ be a $T^*(\mathbb{C}^n)$-modelled GKM graph.

- The graph equivariant cohomology of $\mathcal{G}$ is

$$H^*(\mathcal{G}) = \{\varphi : \mathcal{V} \to H_T^*(pt) \mid \varphi(i(e)) - \varphi(t(e)) \equiv 0 \pmod{\alpha(e)}\}$$

Ring structure of $H^*(\mathcal{G})$

Theorem: Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ set of all hyperplanes in $\mathcal{G}$ satisfying the following

1. $\forall L \in \mathcal{L}$ there exists *unique* pair of *halfspace* H and *its opposite side* $\overline{H}$ s.t $H \cap \overline{H} = L$.
2. $\forall \mathcal{L}' \subseteq \mathcal{L}$ the intersection $\bigcap_{L \in \mathcal{L}'} L$ is either empty or connected.

Let $\mathcal{H} = \{H_1, \ldots, H_m, \overline{H_1}, \ldots, \overline{H_m}\}$ be the set of all halfspaces.
Let $\mathcal{I} = \{\mathcal{H}' \subseteq \mathcal{H} \mid \bigcap_{H \in \mathcal{H}'} H = \emptyset\}$.
Then $H^*(\mathcal{G})$ has the following presentation as $\mathbb{Z}$-algebra

$$\frac{\mathbb{Z}[X, H_1, \ldots, H_m, \overline{H_1}, \ldots, \overline{H_m}]}{\mathcal{I}}$$

where $\mathcal{I}$ is the ideal generated by the following elements

- $H_i + \overline{H_i} - X$ for $1 \leq i \leq m$
- $\prod_{H \in \mathcal{H}'} H$ whenever $\mathcal{H}' \subseteq \mathcal{I}$.

Simplicial complex $\Delta_{\mathcal{L}}$

- A vertex $v_i \in \Delta_{\mathcal{L}} \leftrightarrow L_i \in \mathcal{L}$
- $\langle v_{i_1}, v_{i_2}, \ldots, v_{i_k} \rangle \in \Delta_{\mathcal{L}}$ whenever $L_{i_1} \cap \cdots \cap L_{i_k} \neq \emptyset$
- Consider $\widetilde{\mathcal{G}} = (\Gamma, \tilde{\alpha}, \nabla)$ denote x-forgetful graph associated to $\mathcal{G}$.
- $\mathbb{Z}[\widetilde{\mathcal{G}}] := \mathbb{Z}[L_1, \ldots, L_m]/\mathcal{J}$ where $\mathcal{J}$ is the ideal generated by $L_{i_1} \cdots L_{i_k}$ whenever $L_{i_1} \cap \cdots \cap L_{i_k} = \emptyset$
- If $\Delta_{\mathcal{L}}$ is *a shellable simplicial complex* then we can find a canonical set of monomial generators for $H^*(\widetilde{\mathcal{G}}) \simeq \mathbb{Z}[\widetilde{\mathcal{G}}]$ as a $H^*_{T^n}(pt)$-module.

References

- R. Bielawski and A. Dancer, The geometry and topology of toric hyperKähler manifolds, *Comm. Anal. Geom.* **8** (2000), no. 4, 727–760.
- V. Guillemin, C. Zara, *1-skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J. **107** (2001), 283–349.
- M. Harada, N. Proudfoot, *Properties of the residual circle action on a hypertoric variety*, Pacific J. of Math, Vol. 214, No. 2 (2004), 263–284.
- H. Konno: Cohomology rings of toric hyperKähler manifolds, *Internat. J. Math.***11** (2000), no. 8, 1001–1026.
- H. Maeda, M. Masuda, T. Panov, *Torus graphs and simplicial posets*, Adv. Math. **212** (2007), 458–483.