TOWARD THE CLASSIFICATION OF (REAL) TORIC MANIFOLDS OF PICARD NUMBER 4

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Toric varieties (resp., real toric varieties) are classified by fans (resp., mod 2 fans). More generally, (real) toric spaces can be classified by pairs (K, λ) consisting of a simplicial complex K and a (mod 2) characteristic map λ over K. It is known that the space obtained from a pair (K, λ) is a smooth manifold if the simplicial complex K is a PL-sphere and λ is non-singular.

The classification of smooth (real) toric varieties for Picard number smaller than 4 has been entirely achieved (see [1, 2]), so the next step is Picard number 4. The Picard number of a toric variety over an $(n-1)$-dimensional star-shaped PL-sphere K having m vertices is $m-n$. Thus, it is natural to define the Picard number of K as $\mathrm{Pic}(K) = m - n$. The wedge operation on a vertex of a simplicial complex preserves both the PL-sphereness and the Picard number. Simplicial complexes which cannot be described as the wedge of a lower dimensional simplicial complex are called *seeds*. In [3], Choi and Park have described a way to construct the characteristic maps over any wedged simplicial complex by using a *puzzle* starting from the characteristic maps over its seed. In addition, they also showed there are at most finite seeds supporting a characteristic map. More precisely, in this case when $\mathrm{Pic}\, K = 4$, we have $n \leq 11$. Hence, toward the classification of (real) toric manifolds of Picard number 4, we try to enumerate every seed PL-spheres of Picard number 4 supporting a mod 2 characteristic map as the very first step.

As for the enumeration of seed PL-spheres supporting mod 2 characteristic maps, the classical way was to find all PL-spheres and check the seedness and the existence of mod 2 characteristic maps using the famous Garrison-Scott algorithm. This method leads us to get the complete list up to $n = 6$.

In this work, we additionally develop the linear algebraic method using the condition of supporting mod 2 characteristic functions. This method allowed us to obtain the result for $n = 7$ and to show that the inequality $n \leq 11$ is optimal as presented in the following table.

(n,m)	$(2,6)$	$(3,7)$	$(4,8)$	$(5,9)$	$(6,10)$	$(7,11)$
Number of Seed PLS supporting chr. ftns	1	4	21	142	**733**	**1190**
Number of PLS	1	5	39	337	**6257**	?
Number of Polytopes	1	5	37	322	?	?

(n,m)	$(8,12)$	$(9,13)$	$(10,14)$	$(11,15)$	$(n, n+4)$ with $n > 11$
Number of Seed PLS supporting chr. ftns	> 0	> 0	> 0	$\geq \mathbf{2}$	0

TABLE 1. Data for the dimensions where the results have been obtained.

If time permits, I will present a website associated with a database containing all the known results concerning the topological data of every known PL-spheres.

This is a joint work with Suyoung Choi and Hyuntae Jang.

REFERENCES

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