Toward the enumeration of Picard number 4 (Real) Toric manifolds.

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1. **Backgrounds**
   - Important definitions
   - Theorems

2. **The way to deal with the problem and store the obtained data**
   - How to enumerate every Toric manifolds
   - Database and website

3. **Known and new results**
   - History
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Important definitions

Let $K$ be a simplicial complex on $[m]$.

**Definition 1 (Characteristic map).**

A (non-singular) $\mathbb{Z}_2$-characteristic map over $K$ is a map $\lambda : [m] \to \mathbb{Z}_2^n$. It is non-singular if it satisfies the so-called *non-singularity condition*:

$$\{i_1, \ldots, i_s\} \in K \iff \lambda(i_1), \ldots, \lambda(i_s) \text{ are linearly independant.}$$

$\Lambda(K) = \{\text{Characteristic maps over } K\}$. 

$\mathbb{GL}_n(\mathbb{Z}_2) \curvearrowright \Lambda(K)$, orbits are D-J classes $\text{DJ}(K)$. 

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. 

We find $\bar{\lambda} \in \mathcal{M}_{m,m-n}(\mathbb{Z}_2)$ such that $\lambda \bar{\lambda} = 0$.

**Definition 2 (Dual characteristic map).**

Such $\bar{\lambda}$ defines a dual characteristic map $\bar{\lambda} : [m] \to \mathbb{Z}_2^{m-n}$ over $K$.

A simplicial complex $K$ is said *colorizable* if it supports a (dual) characteristic map.
Proposition 3.

Let $K$ be a simplicial complex on $[m]$ of dimension $n - 1$, $\lambda \in \text{DJ}(K)$, and $\bar{\lambda}$ its dual. Let $J$ be a subset of $[m]$. The following are equivalent:

1. $\bar{\lambda}(J^c)$ is a basis of $\mathbb{Z}_2^{m-n}$;
2. $\lambda(J)$ is a basis of $\mathbb{Z}_2^n$.

So dual characteristic maps and characteristic maps share equivalent data.

Definition 4 (Wedge operation).

Let $K$ be a simplicial complex on $V$ and $p \in V$ being a vertex of $K$. The wedge of $K$ as $p$ is the simplicial complex on $V \cup \{p_1, p_2\}\{p\}$ defined as follows:

$$\text{Wed}_p(K) := (I \ast \text{Lk}_K(p)) \cup (\partial I \ast K\{p\}),$$

where $I$ is the 1-simplex with vertices $\{p_1, p_2\}$, and $K \backslash F := \{\sigma \in K : F \not\subset \sigma\}$, for a face $F \in K$. 

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Simplicial complexes which are not wedges are called *seeds*.
Wedge operation: commutative and associative. We can define a more general wedge operation.

**Definition 5 (Extended wedge operation, Bahri-Benderski-Cohen-Gtiler).**

Let $K$ be a simplicial complex on $[m]$, and $J = (j_1, \ldots, j_m) \in (\mathbb{N}^*)^m$. We define the wedged simplicial complex $K(J)$ as the simplicial complex obtained after performing $j_i - 1$ wedges on the vertex $i$ for $i = 1, \ldots, m$. 

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Remark 6.

Any simplicial complex $L$ which is not a seed can always be represented as a wedged simplicial complex $K(J)$ with $K$ being a seed.

The combinatorial data of a simplicial complex $L$ is a pair $(K, J)$ with $K$ a seed and $J$ an $m$-tuple. But in toric topology, we are working on pairs $(K, \lambda)$.

Question 7.

Is there a constructive way of obtaining $\Lambda(L)$ from $\Lambda(K)$ and $J$?

$$(\Lambda(K), J) \longrightarrow (\Lambda(K(J))).$$
**Definition 8 (projection).**

We define the *projection* of $\lambda$ over $K$ with respect to a vertex $p$ of $K$ as follows:

$$\text{proj}_p(\lambda)(w) := \lambda(w)/\langle\lambda(p)\rangle.$$ 

The projection is a characteristic map on the link of $K$ at the vertex $p$. $\lambda_1$ and $\lambda_2$ are called $p$-adjacent if there exists a CM $\lambda$ over $\text{Wed}_p(K)$ such that $\text{proj}_{p_1}(\lambda) = \lambda_1$ and $\text{proj}_{p_2}(\lambda) = \lambda_2$.

$G(J)$: 1-skeleton of $\Delta^J := \Delta^{j_1-1} \times \ldots \times \Delta^{j_m-1}$

its irreducible cycles are triangles and squares.

**Definition 9 (Puzzle. Choi, Park, 2017).**

A *puzzle* on a wedged simplicial complex $K(J)$, with $K$ on $[m]$ and $J = (j_1, \ldots, j_m)$ is a map $\pi : V(G(J)) \rightarrow \text{DJ}(K)$.

A puzzle is called realizable if the image of the edges, resp. subsquares, of $G(J)$ are $p$-adjacent, resp. realizable.

A realizable puzzle creates a unique D-J class over $K(J)$. 

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Important theorems

- $K$ supports $\lambda \iff \text{Wed}(K)$ supports $\lambda'$;
- $\text{Pic}(K) = \text{Pic}(\text{Wed}(K))$;
- $\bar{\lambda}$ over a seed $\Rightarrow \bar{\lambda}$ is injective (so finite number of seed for a fixed Picard number). Namely, we have $m \leq 2^{\text{Pic}(K)} - 1$ (CHOI-PARK, 2017);
- Puzzle (CHOI-PARK, 2017):

$$\{\text{Realizable puzzles}\} \overset{1:1}{\leftrightarrow} \text{DJ}(K).$$

(the wedge operation is commutative and associative)
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Let $p$ be a fixed Picard number.

The fundamental theorem for toric geometry: toric manifolds are classified by complete non-singular fans.

If a simplicial complex $K$ supports a non-singular fan, then it always supports a mod 2 characteristic map.

The strategy is then to restrict our case to $K$’s which support a mod 2 characteristic map.

<table>
<thead>
<tr>
<th>Direct Garrison-Scott computation</th>
<th>Puzzle method</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Method description</strong></td>
<td><strong>Use the Garrison-Scott algorithm on $K(J)$ directly.</strong></td>
</tr>
</tbody>
</table>
Proposition 10 (Choi-Jang-V, 2021).

The puzzle algorithm is more efficient than the traditional Garisson-Scott algorithm for finding characteristic maps over wedged simplicial complexes.

Thus the last proposition gives us the following methods for finding "every" real toric manifolds of Picard number $p$.

| STEP 1 | Find $CS(p) = \{\text{Colorizable seeds } K \text{ of Pic } p\}$ and DJ($K$) |
|------------------------------------------|
| STEP 2 | Compute $D(K)$ (the characteristic map relation diagram for a puzzle) for every $K \in CS(p)$ |
| STEP 3 | Find the realizable puzzles $\pi : V(G(J)) \rightarrow DJ(K)$. |

Table: The steps of the process.

Remark 11.

- There are infinitely many PL-spheres of Picard number $p$ but any given one can be calculated from this algorithm;
- The finite set $CS(p)$ can be stored in a database for any Picard number $p$. 
See the Website.

The upcoming idea for the website is the following:

- A toric topologist wants to know about a specific simplicial complex $L$;
- She or he visits the website and inputs the maximal faces set of $L$;
- The website finds $L = K(J)$ and uses the puzzle algorithm to find the DJ classes over $K(J)$. 
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Some historical results for (general) toric manifolds:

- Picard number 1 is trivial;
- Picard number 2 (1988, Kleinschmidt) using linear transformations and matroids;
- Picard number 3 (1991, Batyrev based on the work of Kleinschmidt and Sturmfels);

Enumeration of $CS(p)$ for small $p$:

- $CS(1) = \{\partial \Delta^1\}$;
- $CS(2) = \{\partial \Delta^1 \ast \partial \Delta^1\}$;
- $CS(3) = \{\partial \Delta^1 \ast \partial \Delta^1 \ast \partial \Delta^1, P_5, C_4^7\}$.
- $CS(4) = \ldots$
We focus on the STEP 1 of the process: finding all seed PL-spheres and their characteristic maps.

Classic way:

1. Find all PL-spheres (Bistellar move, lexicographic ordering);
2. Select the seeds among them;
3. Use the Garrison-Scott algorithm for finding every characteristic maps over them.

Why is it difficult? Up to $n = 11$, with 15 vertices: number of such simplicial complexes: $2^{(15 \choose 11)} = 2^{1365} \ldots$

Methods B-M or Lexico lowered this complexity but results obtained only up to $n = 6$ (3 months), $n = 7$ unreachable.

<table>
<thead>
<tr>
<th>(n,m)</th>
<th>(2, 6)</th>
<th>(3, 7)</th>
<th>(4, 8)</th>
<th>(5, 9)</th>
<th>(6, 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colorizable seed PLS</td>
<td>1</td>
<td>4</td>
<td>21</td>
<td>142</td>
<td>733</td>
</tr>
<tr>
<td>PLS</td>
<td>1</td>
<td>5</td>
<td>39</td>
<td>337</td>
<td>6257</td>
</tr>
<tr>
<td>Polytopes</td>
<td>1</td>
<td>5</td>
<td>37</td>
<td>322</td>
<td>?</td>
</tr>
</tbody>
</table>

Table: Data for the dimensions where the results have been obtained with the classic methods.
Description of the new method:

1. Restrict the number of IDCM (orbits of the permutation action on the columns);
2. Fix $\bar{\lambda}$, injective in an orbit;
3. Select the maximal faces compatible with $\bar{\lambda}$:
   - the set $\text{MF}(\bar{\lambda}) = \{F_1, \ldots, F_q\}$ (Maximal faces), and
   - $\partial \text{MF}(\bar{\lambda}) = \{f_1, \ldots, f_p\}$ (facets);
4. Use linear algebra (pseudo manifold condition $=$ a facet $f_i$ should be included in exactly two maximal faces):

   Matrix of the (highest dimensional) boundary operator on
   
   $\text{MF}(\bar{\lambda})$: $M = m_{i,j} \in \mathcal{M}_{p,q}(\mathbb{Z}_2)$, with
   
   $m_{i,j} = \begin{cases} 1 & f_i \subset F_j \\ 0 & \text{otherwise} \end{cases}$,

   and $F_i \in \text{MF}(\bar{\lambda})$ and $f_j \in \partial \text{MF}(\bar{\lambda})$.

   We denote by $\mathcal{K}(\bar{\lambda})$ the set of simplicial complexes supporting $\bar{\lambda}$.
   A simplicial complex $K \in \mathcal{K}(\bar{\lambda})$ is a vector in $\mathbb{Z}_2^q$.
   $\mathcal{K}(\bar{\lambda}) \subset \ker_{\mathbb{Z}_2}(M)$.

   Find a basis of the kernel of $M$ (Gaussian elimination) $\rightarrow$ Finite number of linear combinations.
The way to deal with the problem and store the obtained data

The inequality $m \leq 2^{\text{Pic}(K)} - 1$ is optimal for $\text{Pic}(K) = 4$.
Thank you for listening!