Toward the enumeration of Picard number 4 (Real) Toric manifolds.

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Important definitions

Let K be a simplicial complex on $[m]$.

Definition 1 (Characteristic map).

A (non-singular) $\mathbb{Z}_2$-characteristic map over K is a map $\lambda : [m] \to \mathbb{Z}_2^n$. It is non-singular if it satisfies the so-called *non-singularity condition*:

$$\{i_1, \ldots, i_s\} \in K \Leftrightarrow \lambda(i_1), \ldots, \lambda(i_s) \text{ are linearly independant.}$$

$\Lambda(K) = \{\text{Characteristic maps over K}\}$.
$\mathrm{GL}_n(\mathbb{Z}_2) \curvearrowright \Lambda(K)$, orbits are D-J classes $\mathrm{DJ}(K)$.
$\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \end{pmatrix}$.
We find $\bar{\lambda} \in \mathcal{M}_{m,m-n}(\mathbb{Z}_2)$ such that $\lambda\bar{\lambda} = 0$.

Definition 2 (Dual characteristic map).

Such $\bar{\lambda}$ defines a dual characteristic map $\bar{\lambda} : [m] \to \mathbb{Z}_2^{m-n}$ over K.

A simplicial complex K is said *colorizable* if it supports a (dual) characteristic map.

Proposition 3.

Let K be a simplicial complex on $[m]$ of dimension $n-1$, $\lambda \in \mathrm{DJ}(K)$, and $\bar{\lambda}$ its dual. Let J be a subset of $[m]$. The following are equivalent:

1. *$\bar{\lambda}(J^c)$ is a basis of $\mathbb{Z}_2^{m-n}$;*
2. *$\lambda(J)$ is a basis of $\mathbb{Z}_2^n$.*

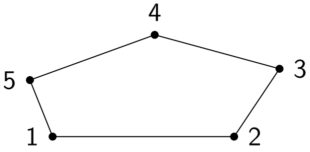
So dual characteristic maps and characteristic maps share equivalent data.

Definition 4 (Wedge operation).

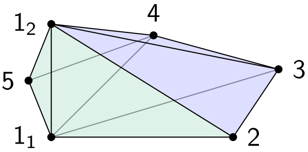
Let K be a simplicial complex on V and $p \in V$ being a vertex of K. The wedge of K as p is the simplicial complex on $V \cup \{p_1, p_2\} \backslash \{p\}$ defined as follows:

$$\mathrm{Wed}_p(K) := (I * \mathrm{Lk}_K(p)) \cup (\partial I * K \backslash \{p\}), \tag{1.1}$$

where I is the 1-simplex with vertices $\{p_1, p_2\}$, and $K \backslash F := \{\sigma \in K : F \not\subset \sigma\}$, for a face $F \in K$.



The pentagon P_5



$\mathrm{Wed}_1(P_5)$

Simplicial complexes which are not wedges are called *seeds*.
Wedge operation: commutative and associative. We can define a more general wedge operation.

Definition 5 (Extended wedge operation,Bahri-Benderski-Cohen-Gtiler).

Let K be a simplicial complex on $[m]$, and $J = (j_1, \dots, j_m) \in (\mathbb{N}^\star)^m$. We define the wedged simplicial complex $K(J)$ as the simplicial complex obtained after performing $j_i - 1$ wedges on the vertex i for $i = 1, \dots, m$.

Remark 6.

Any simplicial complex L which is not a seed can always be represented as a wedged simplicial complex $K(J)$ with K being a seed.

The combinatorial data of a simplicial complex L is a pair (K, J) with K a seed and J an m-tuple.
But in toric topology, we are working on pairs (K, λ).

Question 7.

Is there a constructive way of obtaining $\Lambda(L)$ from $\Lambda(K)$ and J ?

$$(\Lambda(K), J) \longrightarrow (\Lambda(K(J)).$$

Definition 8 (projection).

We define the *projection* of λ over K with respect to a vertex p of K as follows:

$$\mathrm{proj}_p(\lambda)(w) := \lambda(w)/\langle\lambda(p)\rangle.$$

The projection is a characteristic map on the link of K at the vertex p.
λ_1 and λ_2 are called p-adjacent if there exists a CM λ over $\mathrm{Wed}_p(K)$ such that $\mathrm{proj}_{p_1}(\lambda) = \lambda_1$ and $\mathrm{proj}_{p_2}(\lambda) = \lambda_2$.
$G(J)$: 1-skeleton of $\Delta^J := \Delta^{j_1-1} \times \ldots \times \Delta^{j_m-1}$
its irreducible cycles are triangles and squares.

Definition 9 (Puzzle. Choi, Park, 2017).

A *puzzle* on a wedged simplicial complex $K(J)$, with K on $[m]$ and $J = (j_1, \ldots, j_m)$ is a map $\pi : \mathrm{V}(G(J)) \to \mathrm{DJ}(K)$.

A puzzle is called realizable if the image of the edges, resp. subsquares, of $G(J)$ are p-adjacent, resp. realizable.
A realizable puzzle creates a unique D-J class over $K(J)$.

Important theorems

- K supports $\lambda \Leftrightarrow$ Wed(K) supports λ';
- $\mathrm{Pic}(K) = \mathrm{Pic}(\mathit{Wed}(K))$;
- $\bar{\lambda}$ over a seed $\Rightarrow \bar{\lambda}$ is injective (so finite number of seed for a fixed Picard number). Namely, we have $m \leq 2^{\mathrm{Pic}(K)} - 1$ (Choi-Park, 2017);
- Puzzle (Choi-Park, 2017):

$$\{\text{Realizable puzzles}\} \xleftrightarrow{1:1} \mathrm{DJ}(K).$$

(the wedge operation is commutative and associative)

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Let $\mathfrak{p}$ be a fixed Picard number.
The fundamental theorem for toric geometry: toric manifolds are classified by complete non-singular fans.
If a simplicial complex K supports a non-singular fan, then it always supports a mod 2 characteristic map.
The strategy is then to restrict our case to K's which support a mod 2 characteristic map.

	Direct Garrison-Scott computation	Puzzle method
Method description	Use the Garrison-Scott algorithm on $K(J)$ directly.	Use the optimized version of the Garrison-Scott algorithm on the seed K. And use the Puzzle algorithm for getting all the CM on $K(J)$..

Proposition 10 (Choi-Jang-V, 2021).

The puzzle algorithm is more efficient than the traditional Garisson-Scott algorithm for finding characteristic maps over wedged simplicial complexes.

Thus the last proposition gives us the following methods for finding "every" real toric manifolds of Picard number $\mathfrak{p}$.

STEP 1	Find $CS(\mathfrak{p}) = \{$Colorizable seeds K of Pic $\mathfrak{p}\}$ and $\mathrm{DJ}(K)$
STEP 2	Compute $D(K)$ (the characteristic map relation diagram for a puzzle) for every $K \in CS(\mathfrak{p})$
STEP 3	Find the realizable puzzles $\pi : \mathrm{V}(G(J)) \to \mathrm{DJ}(K)$.

Table: The steps of the process.

Remark 11.

- There are infinitely many PL-spheres of Picard number $\mathfrak{p}$ but any given one can be calculated from this algorithm;
- The finite set $CS(\mathfrak{p})$ can be stored in a database for any Picard number $\mathfrak{p}$.

See the Website.

The upcoming idea for the website is the following:

- A toric topologist wants to know about a specific simplicial complex L;
- She or he visits the website and inputs the maximal faces set of L;
- The website finds $L = K(J)$ and uses the puzzle algorithm to find the DJ classes over $K(J)$.

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Some historical results for (general) toric manifolds:

- Picard number 1 is trivial;
- Picard number 2 (1988,Kleinschmidt) using linear transformations and matroids;
- Picard number 3 (1991, Batyrev based on the work of Kleinschmidt and Sturmfels) ;

Enumeration of $CS(\mathfrak{p})$ for small $\mathfrak{p}$:

- $CS(1) = \{\partial\Delta^1\}$;
- $CS(2) = \{\partial\Delta^1 * \partial\Delta^1\}$;
- $CS(3) = \{\partial\Delta^1 * \partial\Delta^1 * \partial\Delta^1, \mathcal{P}_5, C_4^7\}$.
- $CS(4) =$?...

We focus on the STEP 1 of the process: finding all seed PL-spheres and their characteristic maps.

Classic way:

1. Find all PL-spheres (Bistellar move, lexicographic ordering);
2. Select the seeds among them;
3. Use the Garrison-Scott algorithm for finding every characteristic maps over them.

Why is it difficult? Up to $n = 11$, with 15 vertices: number of such simplicial complexes: $2^{\binom{15}{11}} = 2^{1365}$...

Methods B-M or Lexico lowered this complexity but results obtained only up to $n = 6$ (3 months), $n = 7$ unreachable.

(n,m)	$(2,6)$	$(3,7)$	$(4,8)$	$(5,9)$	$(6,10)$
Colorizable seed PLS	1	4	21	142	733
PLS	1	5	39	337	6257
Polytopes	1	5	37	322	?

Table: Data for the dimensions where the results have been obtained with the classic methods.

Description of the new method:

1. Restrict the number of IDCM (orbits of the permutation action on the columns);
2. Fix $\bar{\lambda}$, injective in an orbit;
3. Select the maximal faces compatible with $\bar{\lambda}$:
   - the set $\mathrm{MF}(\bar{\lambda}) = \{F_1, \ldots, F_q\}$ (Maximal faces), and
   - $\partial\,\mathrm{MF}(\bar{\lambda}) = \{f_1, \ldots, f_p\}$ (facets);
4. Use linear algebra (pseudo manifold condition = a facet f_i should be included in exactly two maximal faces):
   Matrix of the (highest dimensional) boundary operator on

   $\mathrm{MF}(\bar{\lambda})$: $M = m_{i,j} \in \mathcal{M}_{p,q}(\mathbb{Z}_2)$, with $m_{i,j} = \begin{cases} 1 & f_i \subset F_j \\ 0 & \text{otherwise} \end{cases}$,

   and $F_i \in \mathrm{MF}(\bar{\lambda})$ and $f_j \in \partial\,\mathrm{MF}(\bar{\lambda})$.
   We denote by $\mathcal{K}(\bar{\lambda})$ the set of simplicial complexes supporting $\bar{\lambda}$.
   A simplicial complex $K \in \mathcal{K}(\bar{\lambda})$ is a vector in $\mathbb{Z}_2^q$.
   $\mathcal{K}(\bar{\lambda}) \subset \ker_{\mathbb{Z}_2}(M)$.
   Find a basis of the kernel of M (Gaussian elimination) $\rightarrow$ Finite number of linear combinations.

(n,m)	$(2,6)$	$(3,7)$	$(4,8)$	$(5,9)$	$(6,10)$	$(7,11)$
Colorizable seed PLS	1	4	21	142	733	1190
PLS	1	5	39	337	6257	?
Polytopes	1	5	37	322	?	?

Table: Data for the dimensions where the results have been obtained.

(n,m)	$(8,12)$	$(9,13)$	$(10,14)$	$(11,15)$
Colorizable seed PLS	≥ 627	≥ 155	≥ 22	≥ 3

Table: Partial results obtained for higher dimensions.

The inequality $m \leq 2^{Pic(K)} - 1$ is optimal for $\text{Pic}(K) = 4$.

Thank you for listening !