Orbifold Stiefel-Whitney classes on right-angled Coxeter complexes

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Content

1. Right-angled Coxeter complexes and its homology groups

2. Orbifold Stiefel-Whitney classes on RACC

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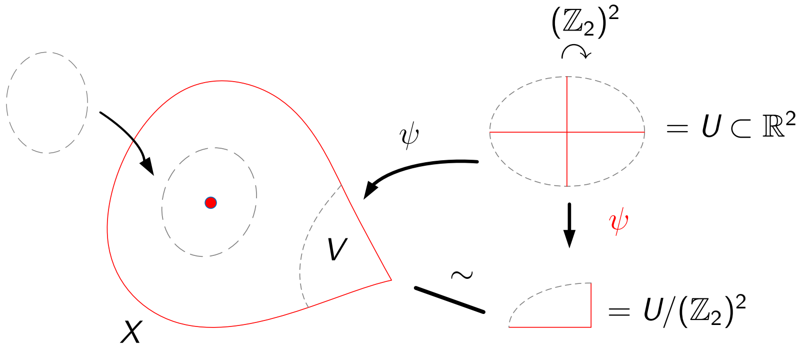

♣ A *right-angled Coxeter n-orbifold* (RACO) is an orbifold locally modelled on $\mathbb{R}^n/(\mathbb{Z}_2)^k$, where $(\mathbb{Z}_2)^k$-action is generated by standard reflections (across the coordinate hyperplanes).

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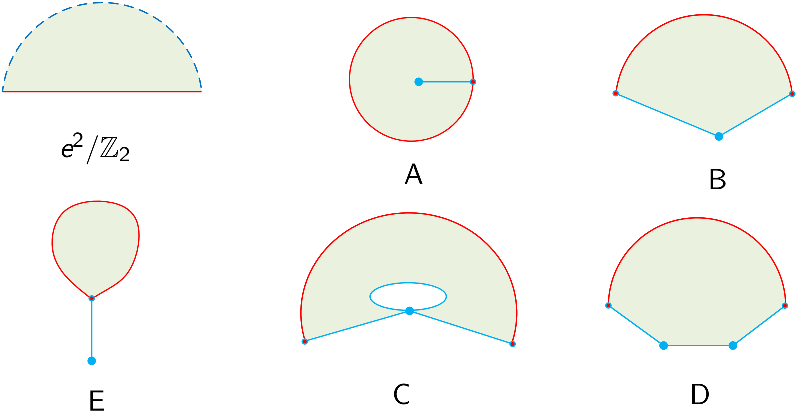
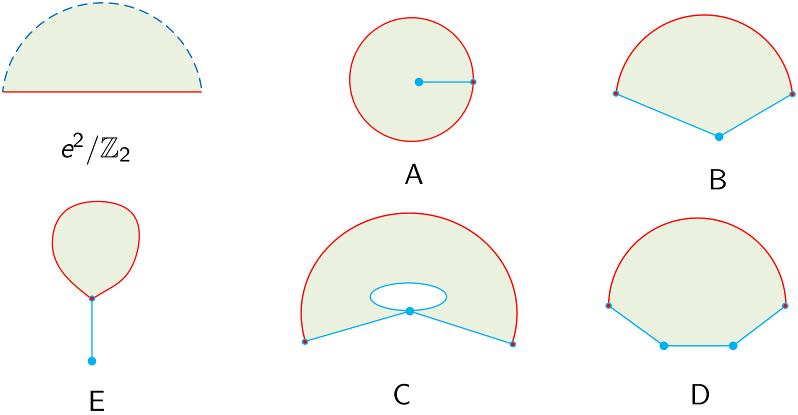
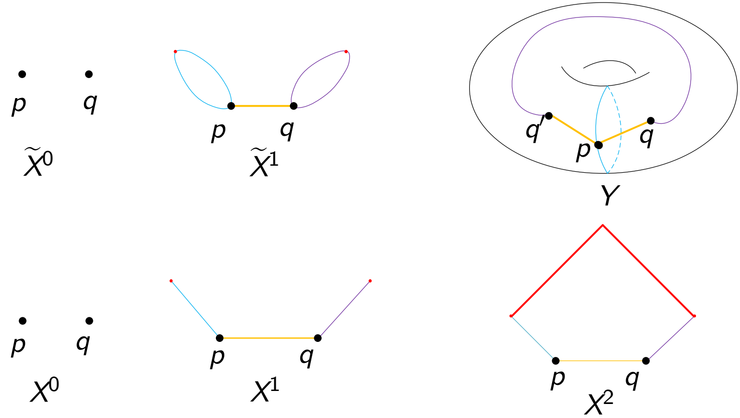
♣ Facial structure.

An example

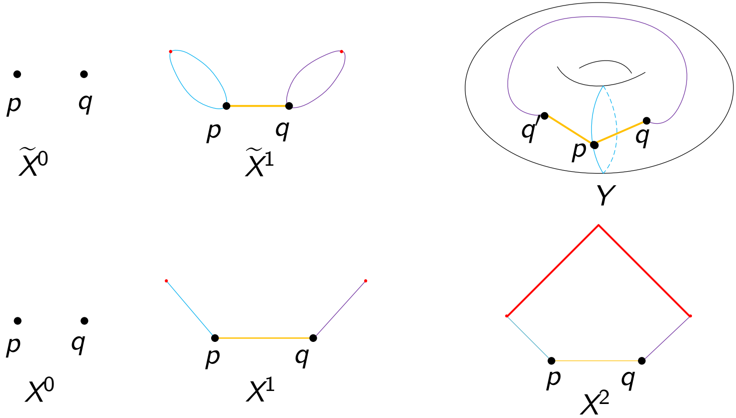




$(\star)$ E is not a RACC.



p q
$\widetilde{X}^0$
p q
$\widetilde{X}^1$
q' p q
Y
p q
X^0
p q
X^1
p q
X^2



$(\star)$ $\widetilde{X}^2 = Y/[p, q] \sim [p, q'] \simeq T^2$.

Boundary maps of RACC are defined by boundary maps of its blow-up:

$$d(e^n/W) = \sum n_\beta \left(\frac{|W|}{|W_\beta|} \bmod 2 \right) e_\beta^{n-1}/W_\beta$$

where W_β is a subgroup of W.

Boundary map and orbifold homology groups

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Theorem (Lü-Wu-Yu)

Let X be an n-dimensional right-angled Coxeter complex, and T be the face set of X. Then

$$H_i^{orb}(X) = \bigoplus_{f \in T} H_{i-l(f)}(X_J) \tag{1}$$

where $l(f)$ is the local codimension of f.

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$$H^*_{orb}(X) := H^*(\widetilde{X}).$$

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Example (Simple polytope)

Let P be a simple polytope equipped with a right-angled Coxeter orbifold structure. Then the standard cubical decomposition of P is a RACC. Then

$$H_i^{orb}(P) = \mathbb{Z}^{f_{n-i}} \tag{2}$$

where f_{n-i} is the number of $(n-i)$-faces of P.

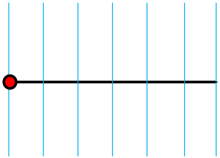
$$H^*_{orb}(P; \mathbb{Z}_2) = \mathbb{Z}_2[v_1, \cdots, v_m]/I + J$$

where I is the Stanley-Reisner ideal of P, $J = (v_i^2, \forall i)$.

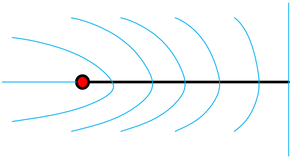
Orbifold vector bundle, Satake

E, X be two orbifolds with orbifold structures $\{U^*, \psi^*, G^*\}$ and $\{U, \psi, G\}$, an orbifold vector bundle $\pi : E \to X$ satisfies:

- $$\begin{tikzcd}
E \arrow[d] & E_{U/G} \arrow[d, "\pi_H"'] & U^* \cong U \times \mathbb{R}^m \arrow[l, "\psi_{U^*}"'] \arrow[d, "\widetilde{\pi}"'] & \curvearrowleft G^* \\
X & U/G & U \arrow[l, "\psi_U"] & \curvearrowleft G
\end{tikzcd}$$
- Compatibility conditions.

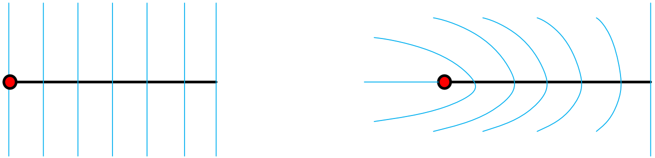


$\overline{E}$



$\widetilde{E}$

$$\overline{E} = D^1 \times \mathbb{R}/(x,y) \sim (-x,y)$$
$$\widetilde{E} = D^1 \times \mathbb{R}/(x,y) \sim (-x,-y)$$



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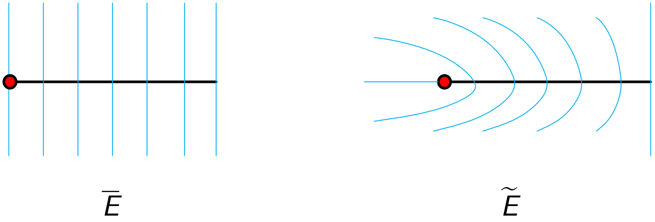

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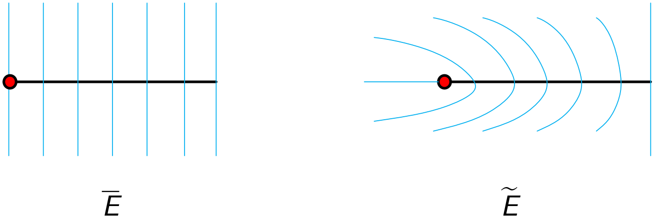
$$w(\overline{E}) := 1 \quad w(\widetilde{E}) := 1+s$$



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(Orbifold verctor bundles over a RACC maybe not a RACC.)

Orbifold SW classes on $D^n/(\mathbb{Z}_2)^k$

In general, for $\pi : E \to D^n/(\mathbb{Z}_2)^k$, there is a representation

$$\rho : (\mathbb{Z}_2)^k \longrightarrow GL_m(\mathbb{R}).$$

The image of ρ on the generator set of $(\mathbb{Z}_2)^k$ gives a $m \times k$ matrix with elements ± 1.

$$\begin{array}{cc} & \begin{array}{cccc} \rho(s_1) & \rho(s_2) & \cdots & \rho(s_k) \end{array} \\ C = & \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^k \\ x_2^1 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^k \end{pmatrix}_{m\times k} \end{array}$$

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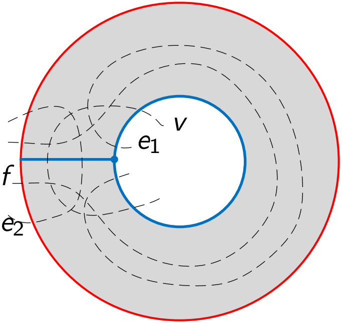
The total orbifold Stiefel-Whitney class of $\pi : E \longrightarrow D^n/(\mathbb{Z}_2)^k$ is defined:

$$w(E) = \prod_{i=1}^{m} \left(1 + \sum_{j=1}^{k} \frac{1 - x_i^j}{2} s_j\right) \in H^*(D^n/(\mathbb{Z}_2)^k; \mathbb{Z}_2). \tag{3}$$



$$H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i=1 \\ \mathbb{Z}_2, & i=2 \\ 0, & \textit{otherwise.} \end{cases}$$

Figure: $X = S^1 \times [-1,1]/\mathbb{Z}_2$

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$$\mathcal{R}_H < H^*(H; \mathbb{Z}_2) < H^*(X; \mathbb{Z}_2)$$

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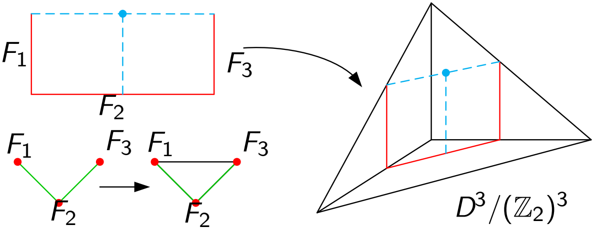
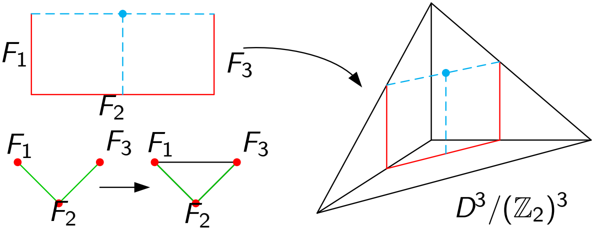


Orbifold SW classes on RACC

$$\pi : E \longrightarrow X \Rightarrow \pi_H : E_H \longrightarrow H \Rightarrow \text{matrix } C_H \Rightarrow \widetilde{\pi} : E_{C_H} \longrightarrow D^\eta/(\mathbb{Z}_2)^\eta.$$



$$\begin{tikzcd}
E_H \arrow[r] \arrow[d, "\pi_H"'] & E_{C_H} \arrow[d, "\widetilde{\pi}"'] \\
H \arrow[r, "j"] & D^\eta/(\mathbb{Z}_2)^\eta
\end{tikzcd}$$

In general, let $j : \mathcal{N}(H) \to \mathcal{N}(D^\eta/(\mathbb{Z}_2)^\eta) = \Delta^{\eta-1}$ be a simplicial map.

$$j^* : \mathbb{Z}_2[s_1, \cdots, s_\eta]/(s_i^2, \forall i) \longrightarrow \mathcal{R}_H < H^*(X; \mathbb{Z}_2).$$

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Total orbifold SW class of $\pi_H : E_H \longrightarrow X_H$ is defined:

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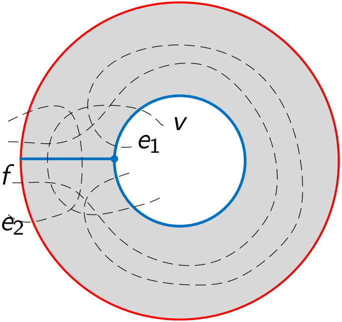
Total orbifold SW class of $\pi : E \longrightarrow X$ is defined:

$$w(E) = w(E(X_{reg})) \cdot \prod_H w(E_H). \tag{6}$$

where $E(X_{reg}) = \pi|_{X_{reg}}$.

Example



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$$w(TX) = w(TS^1)(1+s) = 1+s.$$

Figure: $X = S^1 \times [-1,1]/\mathbb{Z}_2$

Thank You

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