Toric Topology 2021

in Osaka (Online)

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On the sign ambiguity in the equivariant cohomological rigidity for GKM graphs

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Toric topology 2021 in Osaka (Zoom)

Slide :

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Content

(1) GKM graph

(2) Equivariant rigidity

(3) Sign ambiguity in equivariant rigidity

(4) Equivariant total Chern class

(5) Main result

Torus equivariant cohomology

- T : compact torus of rank r ($T \cong (S^1)^r$ as Lie groups)

- X : T-space

T-equivariant cohomology

- $H_T^*(X) := H^*(ET \times_T X)$. ← $\mathbb{Z}$-coefficient

$S^\infty = \varinjlim (S^1 \hookrightarrow S^3 \hookrightarrow S^5 \hookrightarrow \cdots)$

- $X_1 := \{ x \in X \mid \dim T_x \geqq r-1 \}$

equivariant 1-skeleton

GKM theory

Ref

(1) Franz - Puppe, Exact cohomology sequences with Integral coefficients for torus actions (Transform. Groups 12 (2007))

(2) ———————, Exact sequences for equivariantly formal spaces (C.R. Math. Acad. Sci. Soc. R. Can. 33 (2011))

If X is an equivariantly formal GKM manifold, one can describe the image of the Borel localization

$$\boxed{H_T^*(X) \rightarrowtail H_T^*(X^T)}.$$

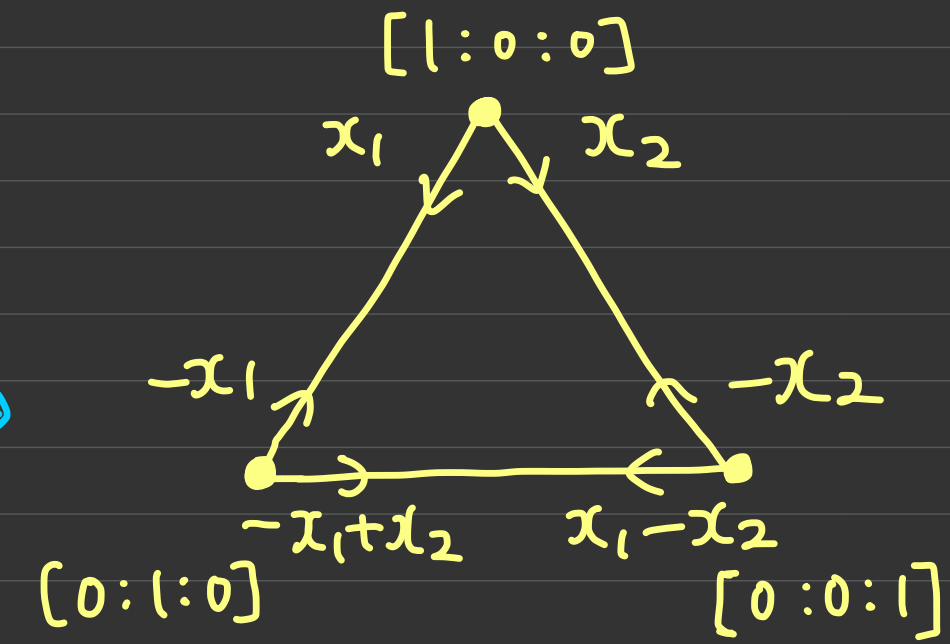
$\boxed{\text{Example}}$

① $(r,n) = (2,2)$

$X = \mathbb{C}P^2$

$(t_1, t_2) \cdot [z_0 : z_1 : z_2]$

$:= [z_0, t_1 z_1, t_2 z_2]$

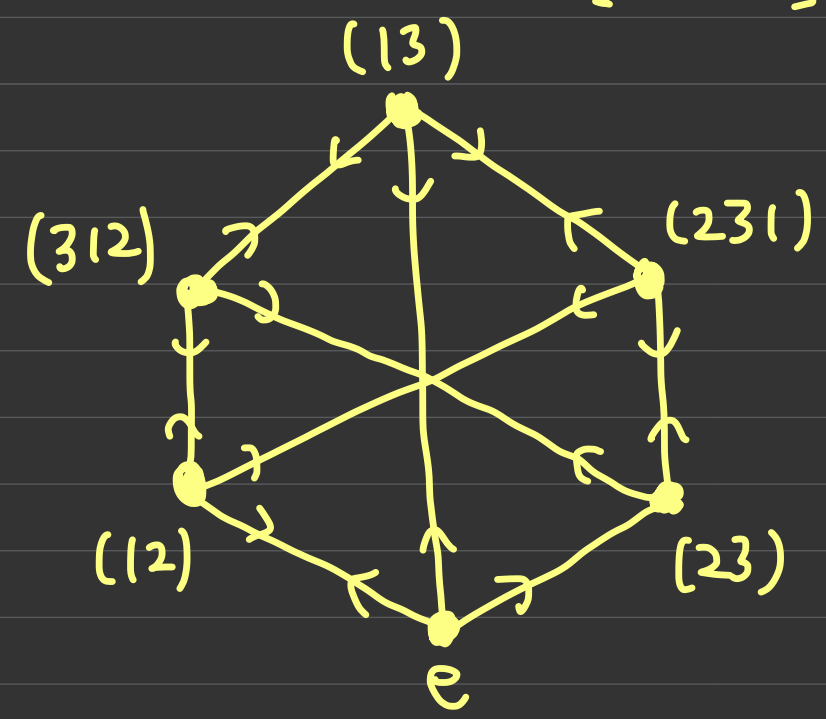


② $(r,n) = (2,3)$

$X = Fl(\mathbb{C}^3)$

$t \cdot [V_0 \subset V_1 \subset V_2 \subset V_3]$

$:= [tV_0 \subset tV_1 \subset tV_2 \subset tV_3]$



③ GKM manifold X $\rightsquigarrow$ GKM graph $\mathcal{G}_X$

Franz-Puppe's condition

For any $x \in X$, the stabilizer T_x is connected or T/T_x is a cyclic group of finite order

e.g.

- $(\mathbb{C}^*)^n \curvearrowright$ (n-dim toric variety)
- $(S^1)^n \curvearrowright$ ($2n$-dim sphere)

Equivariant formality

$H_T^*(X)$ is free / $H^*(BT)$

e.g.

- Spaces having T-invariant affine pavings

GKM localization / $\mathbb{Z}$

$\mathbb{Z}$-coeffient

Thm (Goresky-Kottwitz-MacPherson, Franz-Puppe)

If a GKM manifold X satisfies

① Franz-Puppe's condition and

② Equivariant Formality

(the image of the Borel localization)

$$H_T^*(\mathcal{G}_X) := \left\{ f \in H^*(BT)^{\nu} \;\middle|\; \begin{array}{l} \alpha(e) \mid (f(p) - f(q)) \\ \text{if } e \text{ connects } p \,\&\, q \end{array} \right\}.$$

Graph equivariant cohomology

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Abstract GKM graph

Guillemin - Zara introduced the notion of an abstract GKM graph.

Notation

- $\mathcal{G} := (\mathcal{V}, \mathcal{E})$: n-valent graph

  vertex

  oriented edge

- $\mathcal{E} \xrightarrow{\alpha} H^2(BT)$: a map from $\mathcal{E}$ to $H^2(BT)$

  $H^2(BT) \cong \mathbb{Z}[x_1, \cdots, x_r]$, $\deg x_i = 2$

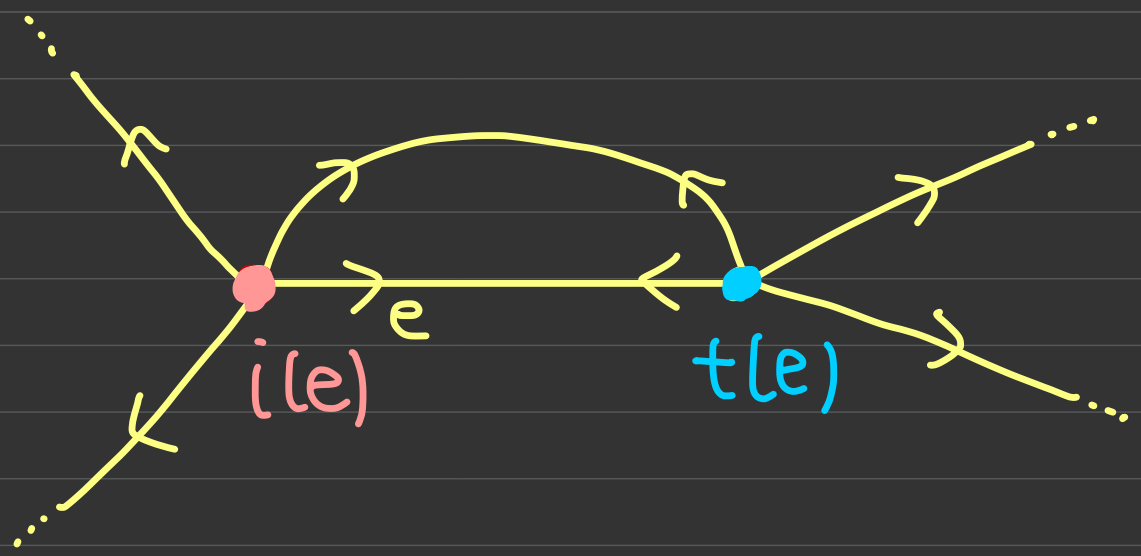
  s.t. $\alpha(\bar{e}) = \pm \alpha(e)$

  orientation-reversed

► $\nabla = \{\nabla_e\}_{e \in \mathcal{E}}$: a family of bijections

$$\nabla_e : \mathcal{E}_{\underbrace{i(e)}_{\text{initial point}}} \longrightarrow \mathcal{E}_{\underbrace{t(e)}_{\text{termial point}}}$$

where $\mathcal{E}_* = \{ e \in \mathcal{E} \mid i(e) = * \}$.



$\boxed{\text{Def}}$

The pair $(\mathcal{G}, \alpha)$ is called an **abstract GKM graph** of type (r, n) if there exist at least one ∇ s.t.

① $\nabla_e(e) = \bar{e}$

② $\alpha(\nabla_e(e')) - \alpha(e') \in \mathbb{Z}\,\alpha(e)$

$(\forall e' \in \mathcal{E}_{i(e)})$.

∇ is called a **connection** of $(\mathcal{G}, \alpha)$.

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$\boxed{\text{Def}}$ (Franz - Y. 2019)

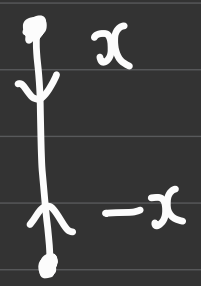
Two GKM graphs $(\mathcal{G}, \alpha)$, $(\mathcal{G}', \alpha')$ are **isomorphic**

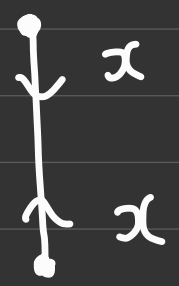
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ graph isomorphism

$$\varphi_0 : \mathcal{V} \xrightarrow{\sim} \mathcal{V}', \quad \varphi_1 : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$$

s.t. $\alpha(\varphi_1(e)) = \pm\, \alpha(e)$ //

<u>e.g.</u> $(r, n) = (1, 2)$

$\mathcal{G} =$ 

$\underset{\text{isom}}{\cong}$ $\mathcal{G}' =$ 

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In the rest of this talk, we assume that

the GCD of coefficients of $\alpha(e)$ is 1 for any $e \in \mathcal{E}$.

(Recall : $H^*(BT) \cong \mathbb{Z}[x_1, \cdots, x_r]$)

- This assumption reflects Franz-Puppe's condition on stabilizers.

Equivariant rigidity

Let $(\mathcal{G}, \alpha)$ and $(\mathcal{G}', \alpha')$ be GKM graphs of type (r, n).

$\boxed{\text{Thm}}$ (Franz - Y. (2019))

$H_T^*(\mathcal{G}) \cong H_T^*(\mathcal{G}')$ as graded $H^*(BT)$-algebras

$\Longleftrightarrow$ $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic.

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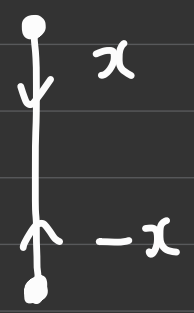
- Motivated by Masuda's rigidity result on toric manifolds.

Sign ambiguity

- In general, one can not remove the sign in the equality

$$\alpha(\varphi_1(e)) = \pm\, \alpha(e)$$

e.g.

$\mathcal{G} =$ 

& $\mathcal{G}' =$ 

have the same graph equivariant cohomology.

Question

How can we recover GKM graphs from its graph equivariant cohomology?

An answer

The notion of

"equivariant total Chern classes"

resolves the sign ambiguity.

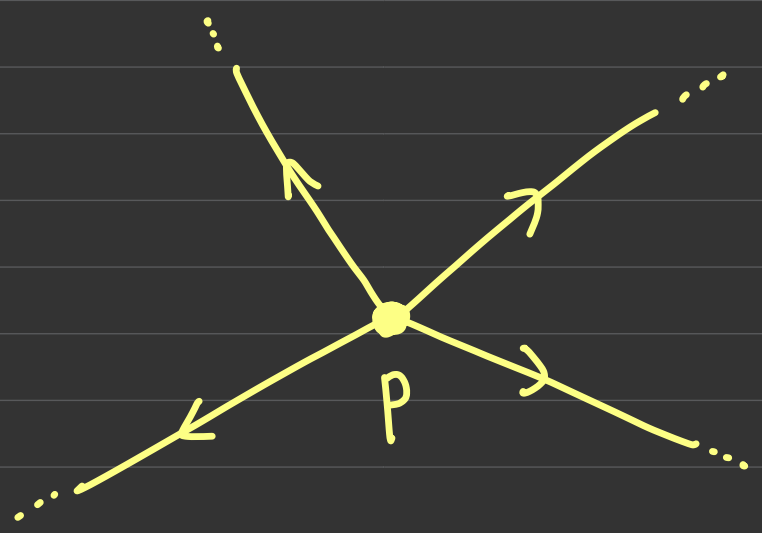
Construction

Step 1

For each $p \in \mathcal{V}$, we give an ordering

$$\alpha_{p,1}, \cdots, \alpha_{p,n}$$

of elements in $\{\alpha(e) \mid i(e) = p\}$.



$\boxed{\text{Step 2}}$

- Let $P \in \mathbb{Z}[y_1, \cdots, y_n]^{\mathfrak{S}_n}$ be a symmetric polynomial with n-variables.

- $P(\alpha_{p,1}, \cdots, \alpha_{p,n}) \in H^*(BT)$ is independent of the ordering on $\{\alpha(e) \mid i(e) = p\}$.

$\boxed{\text{Step 3}}$

$\boxed{\text{Lem}}$ (Y.)

The map

$$f_P : \mathcal{V} \longrightarrow H^*(BT)$$
$$P \longmapsto P(\alpha_{P,1}, \cdots, \alpha_{P,n})$$

provides an element of $H^*_T(\mathfrak{g})$. //

► For its proof, one needs the existence of a connection.

$\boxed{\text{Step 4}}$

$\boxed{\text{Def}}$ (Y.)

The equivariant total Chern class $c^T(\mathcal{G})$ of $\mathcal{G}$ is the map

$$\mathcal{V} \longrightarrow H^*(BT)$$

associated with the symmetric polynomial

$$P(y_1, \cdots, y_n) = \prod_{i=1}^{n} (1 + y_i).$$

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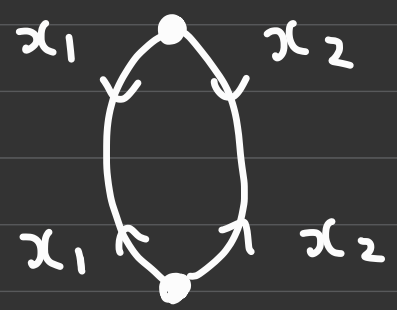
$\boxed{\text{Remark}}$

- Some GKM graphs naturally arise from non almost complex manifold.

(e.g. $\mathcal{G} :=$  is a GKM graph of type $(2,4)$ which arises from S^4.)

- One can define $c^T(\mathcal{G})$ for such a GKM graph.

Main result

Def

Two GKM graphs $(\mathcal{G}, \alpha)$, $(\mathcal{G}', \alpha')$ are **geometrically isomorphic**

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ graph isomorphism

$$\varphi_0 : \mathcal{V} \xrightarrow{\sim} \mathcal{V}', \quad \varphi_1 : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$$

s.t. $\alpha(\varphi_1(e)) = \alpha(e)$. //

(Recall : isomorphic $\longleftrightarrow$ $\alpha(\varphi_1(e)) = \pm\, \alpha(e)$)

Let $(\mathcal{G}, \alpha)$ and $(\mathcal{G}', \alpha')$ be GKM graphs of type (r, n).

$\boxed{\text{Thm}}$ (Y.)

$\exists$ graded $H^*(BT)$-algebra isomorphism

$$H_T^*(\mathcal{G}) \longrightarrow H_T^*(\mathcal{G}')$$

preserving c^T

$\Longleftrightarrow$ $\mathcal{G}$ and $\mathcal{G}'$ are geometrically isomorphic.

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$\boxed{\text{Remark}}$

- One can show more:
  Such an isomorphism
  $$H_T^*(\mathcal{G}) \longrightarrow H_T^*(\mathcal{G}')$$
  is induced by a geometric isomorphism.
  (strong rigidity)

- Thanks to the lemma, one can define a ring homomorphism

$$H^*(BT)[y_1, \cdots, y_n]^{\mathfrak{S}_n} \longrightarrow H_T^*(\mathcal{Y}).$$

- Since $\mathbb{Z}[y_1, \cdots, y_n]^{\mathfrak{S}_n}$ is generated by elementary symmetric polynomials,

$\exists\ H_T^*(\mathcal{Y}) \xrightarrow{\sim} H_T^*(\mathcal{Y}')$ preserving c^T

$\Longleftrightarrow\ \exists\ H_T^*(\mathcal{Y}) \xrightarrow{\sim} H_T^*(\mathcal{Y}')$ of

graded $H^*(BT)[y_1, \cdots, y_n]^{\mathfrak{S}_n}$-algebras

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Digression

- Torus equivariant cohomology frequently behaves like a space.

- Our theorem indicates that it is natural to consider the pair

$$\left(H_T^*(\mathcal{G}),\ c^T(\mathcal{G})\right)$$

as a space with geometric structure (like a symplectic manifold (X, ω)).

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Torus graphs

Following Maeda-Masuda-Panov (Adv. 2007) we consider torus graphs :

Def

A GKM graph of type (n,n) is called a torus graph if for each $p \in V$

$$\{\alpha(e) \mid i(e) = p\}$$

forms a $\mathbb{Z}$-basis of $H^2(BT)$.

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- The GKM graph of a toric manifold is a torus graph.

- In this restricted case, we need only c_1^T (= degree 2 part of c^T).

Let (T, α) and (T', α') be torus graphs of type (n, n)

$\boxed{\text{Thm}}$ (Y.)

$\exists$ graded $H^*(BT)$-algebra isomorphism

$$H_T^*(T) \longrightarrow H_T^*(T')$$

preserving c_1^T

$\Longleftrightarrow$ T and T' are geometrically isomorphic.

Summery

1. GKM graphs and its graph equivariant cohomology
2. Sign ambiguity in equivariant rigidity
3. Definition of torus equivariant Chern classes.
4. Resolving the sign ambiguity via $c^T(\mathcal{G})$