Toric Topology 2021

in Osaka (Online)

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On the sign ambiguity in the equivariant cohomological rigidity for GKM graphs

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Toric topology 2021 in Osaka (Zoom)

Slide:

https://researchmap.jp/AFT
(1) GKM graph

(2) Equivariant rigidity

(3) Sign ambiguity in equivariant rigidity

(4) Equivariant total Chern class

(5) Main result
Torus equivariant cohomology

- $T$: compact torus of rank $r$ ($T \cong (S^1)^r$ as Lie groups)

- $X$: $T$-space

- $H_T^*(X) := H^*(ET \times_T X)$, $\mathbb{Z}$-coefficient

- $X_1 := \{ x \in X \mid \dim T_x \geq r-1 \}$

  equivariant $1$-skeleton
GKM theory

Ref

(1) Franz - Puppe, Exact cohomology sequences with Integral coefficients for torus actions (Transform. Groups 12 (2007))


If $X$ is an equivariantly formal GKM manifold, one can describe the image of the Borel localization

$$H^*_T(X) \rightarrow H^*_T(X^T).$$
Example

1. \((r, n) = (2, 2)\)
\[
X = \mathbb{CP}^2
\]
\[
(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0, t_1 z_1, t_2 z_2]
\]

2. \((r, n) = (2, 3)\)
\[
X = \text{Fl}(\mathbb{C}^3)
\]
\[
t \cdot [V_0 \subset V_1 \subset V_2 \subset V_3] = [tV_0 \subset tV_1 \subset tV_2 \subset tV_3]
\]

3. GKM manifold \(X\) \(\xrightarrow{\sim}\) GKM graph \(G_X\)
Franz-Puppe's condition

For any $x \in X$, the stabilizer $T_x$ is connected or $T/T_x$ is a cyclic group of finite order.

e.g.

- $(\mathbb{C}^*)^n \cap (n\text{-dim toric variety})$
- $(S^1)^n \cap (2n\text{-dim sphere})$

Equivariant formality

$H^*_T(X)$ is free / $H^*(BT)$

e.g.

- Spaces having $T$-invariant affine pavings
Thm (Goresky-Kottwitz-MacPherson, Franz-Puppe)

If a GKM manifold $X$ satisfies

1. Franz-Puppe’s condition and
2. Equivariant Formality

\[ \mathcal{H}_T^*(G_x) := \left\{ f \in H^*(B T) \mid \forall e \in (f(p) - f(q)) \text{ if } e \text{ connects } p \& q \right\} \]

Graph equivariant cohomology
Abstract GKM graph

Guillemin-Zara introduced the notion of an abstract GKM graph.

Notation

\[ G := (V, E) : n\text{-valent graph} \]

\[ V \rightarrow \text{vertex} \quad E \rightarrow \text{oriented edge} \]

\[ E \xrightarrow{\lambda} H^2(B\Gamma) : \text{a map from } E \text{ to } H^2(B\Gamma) \]

s.t. \( \lambda(\overline{e}) = \pm \lambda(e) \)

orientation-reversed

\[ \mathbb{Z}[x_1, \ldots, x_r] \]

\[ \deg x_i = 2 \]
\[ \nabla = \{ \nabla_e \}_{e \in E} : \text{a family of bijections} \]

\[ \nabla_e : E_{i(e)} \longrightarrow E_{t(e)} \]

initial point \quad terminal point

where \[ E_\star = \{ e \in E \mid i(e) = \star \} \].
The pair \((G, \lambda)\) is called an abstract GKM graph of type \((r, n)\) if there exist at least one \(\nabla\) s.t.

1. \(\nabla_e(e) = \overline{e}\)

2. \(\lambda(\nabla_e(e')) - \lambda(e') \in \mathbb{Z} \lambda(e)\)
   \((\forall e' \in \mathcal{E}_i(e))\).

\(\nabla\) is called a connection of \((G, \lambda)\).
**Def (Franz-Y. 2019)**

Two GKM graphs \((g, \lambda), (g', \lambda')\) are isomorphic

\[
\exists \text{ graph isomorphism } \\
\varphi_0 : V \xrightarrow{\sim} V', \quad \varphi_1 : E \xrightarrow{\sim} E' \\
\text{s.t. } \lambda'(\varphi_1(e)) = \pm \lambda(e)
\]

**Def**

\[ e.g. (r, n) = (1, 2) \]

\[
g = \begin{array}{c}
\downarrow x \\
\downarrow -x \\
\end{array} \xrightarrow{\text{isom}} \begin{array}{c}
\downarrow x \\
\end{array} \\
\begin{array}{c}
g' = \begin{array}{c}
\downarrow x \\
\end{array} \\
\end{array}
\]
In the rest of this talk, we assume that the GCD of coefficients of $d(e)$ is 1 for any $e \in E$.

(Recall: $H^*(BT) \cong \mathbb{Z}[x_1, \ldots, x_r]$)

This assumption reflects Franz-Puppe’s condition on stabilizers.
Equivariant rigidity

Let $(\mathcal{G}, \mathcal{A})$ and $(\mathcal{G}', \mathcal{A}')$ be GKM graphs of type $(r, n)$.

**Thm** (Franz - Y. (2019))

\[ H^*_T(\mathcal{G}) \cong H^*_T(\mathcal{G}') \text{ as graded } H^*(BT) - \text{algebras} \]

\[ \iff \mathcal{G} \text{ and } \mathcal{G}' \text{ are isomorphic.} \]

Motivated by Masuda's rigidity result on toric manifolds.
Sign ambiguity

In general, one cannot remove the sign in the equality

$$\lambda(\mathcal{P}_1(e)) = \pm \lambda(e)$$

e.g.

$$g = \begin{array}{c}
\begin{bmatrix}
  x \\
-1
\end{bmatrix}
\end{array} \quad \text{&} \quad g' = \begin{array}{c}
\begin{bmatrix}
  x \\
  1
\end{bmatrix}
\end{array}$$

have the same graph equivariant cohomology.
Question

How can we recover GKM graphs from its graph equivariant cohomology?
An answer

The notion of “equivariant total Chern classes” resolves the sign ambiguity.
Construction

Step 1

For each $p \in V$, we give an ordering $d p, 1, \ldots, d p, n$ of elements in $\{d(e) \mid i(e) = p\}$.
Step 2

Let $P \in \mathbb{Z}[y_1, \ldots, y_n]^{\mathfrak{S}_n}$ be a symmetric polynomial with $n$-variables.

$P(\alpha_1, \ldots, \alpha_n) \in H^*(BT)$ is independent of the ordering on $\{ \alpha(e) \mid i(e) = p \}$. 
Step 3

Lemma (Y.)

The map

\[ f_p : V \rightarrow H^*_T(B_T) \]

\[ p \mapsto P(\alpha_p^1, \ldots, \alpha_p^n) \]

provides an element of \( H^*_T(G) \).

For its proof, one needs the existence of a connection.
Step 4

Def (Y.)

The equivariant total Chern class $c^T(G)$ of $G$ is the map

$$
V \rightarrow H^*(BT)
$$

associated with the symmetric polynomial

$$
P(y_1, \ldots, y_n) = \prod_{i=1}^{n} (1 + y_i).
$$
Some GKM graphs naturally arise from non-almost complex manifold. For example, \( G := \begin{array}{ccc} x_1 & \rightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \rightarrow & x_2 \end{array} \) is a GKM graph of type \((2,4)\) which arises from \(S^4\).

One can define \(C^*(G)\) for such a GKM graph.
Main result

Def

Two GKM graphs \((G, \lambda), (G', \lambda')\) are geometrically isomorphic

def \(\iff\) graph isomorphism

\[\psi_0 : V \cong V', \quad \psi_1 : E \cong E'\]

s.t. \(\lambda(\psi_1(e)) = \lambda(e)\).

(Recall: isomorphic \(\iff\) \(\lambda(\psi_1(e)) = \pm \lambda(e)\))
Let \((G, \mathcal{L})\) and \((G', \mathcal{L}')\) be GKM graphs of type \((r, n)\).

**Thm** (Y.)

\[
\exists \text{ graded } H^*(BT)\text{-algebra isomorphism } \\
H^*_T(G) \to H^*_T(G')
\]

preserving \(c^T\)

\(\iff G\) and \(G'\) are geometrically isomorphic.
Remark

One can show more:

Such an isomorphism

\[ H^*_L(\mathcal{G}) \rightarrow H^*_L(\mathcal{G}') \]

is induced by a geometric isomorphism.

(strong rigidity)
Thanks to the lemma, one can define a ring homomorphism

\[ H^*(BT) [y_1, \ldots, y_n] \cong^n \rightarrow H^*_T(g). \]

Since \( \mathbb{Z}[y_1, \ldots, y_n] \cong^n \) is generated by elementary symmetric polynomials,

\[ \exists \ H^*_T(g) \cong H^*_T(g') \text{ preserving } c^T \]

\[ \Leftrightarrow \exists \ H^*_T(g) \cong H^*_T(g') \text{ of graded } H^*(BT) [y_1, \ldots, y_n] \cong^n - \text{algebras} \]
Digression

- Torus equivariant cohomology frequently behaves like a space.

- Our theorem indicates that it is natural to consider the pair $\left( H^*_T(G), c^T(G) \right)$ as a space with geometric structure (like a symplectic manifold $(X, \omega)$).
Torus graphs

Following Maeda–Masuda–Panov (Adv. 2007) we consider torus graphs:

Def

A GKM graph of type $(n,n)$ is called a torus graph if for each $p \in V$

\[
\{ d(e) \mid i(e) = p \}
\]

forms a $\mathbb{Z}$-basis of $H^2(B\mathcal{T})$. //
The GKM graph of a toric manifold is a torus graph.

In this restricted case, we need only $c_1^T (= \text{degree 2 part of } c^T)$. 
Let \((T, L)\) and \((T', L')\) be torus graphs of type \((n, n)\).

\[
\text{Thm} \ (Y.): \exists \text{ graded } H^* (BT)-\text{algebra isomorphism}
\]

\[
H^*_T (T) \longrightarrow H^*_T (T')
\]

preserving \( C^T_1 \)

\( \iff \) \( T \) and \( T' \) are geometrically isomorphic.
Summary

1. GKM graphs and its graph equivariant cohomology
2. Sign ambiguity in equivariant rigidity
3. Definition of torus equivariant Chern classes.
4. Resolving the sign ambiguity via $C^r(G)$