On Fano and weak Fano regular semisimple Hessenberg varieties

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March 24, 2021

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The definition of full flag variety

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- $GL_n(\mathbb{C})$ acts on $\mathrm{Flag}(\mathbb{C}^n)$ transitively and the isotropy group B of the point

$$0 \subset (e_1) \subset (e_1, e_2) \subset \cdots \subset (e_1, e_2, \cdots, e_n)$$

  consists of upper triangular invertible matrices. Hence $\mathrm{Flag}(\mathbb{C}^n)$ is a nonsingular projective variety.

Torus action on $\mathrm{Flag}(\mathbb{C}^n)$

- $$GL_n(\mathbb{C})/B \cong \mathrm{Flag}(\mathbb{C}^n).$$

$$gB \mapsto 0 \subseteq \langle g_1 \rangle \subseteq \langle g_1, g_2 \rangle \subseteq \cdots \subseteq \langle g_1, g_2, \cdots, g_n \rangle.$$

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Hessenberg functions

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Express Hessenberg function by staircase diagram

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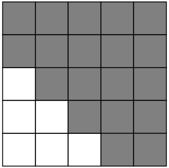

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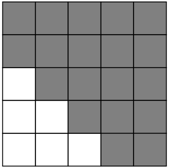

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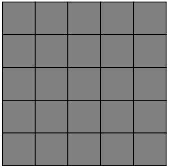
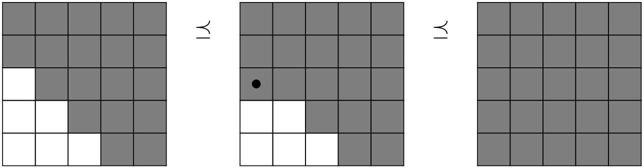


Partial order on Hessenberg functions

Let $h, h' : [n] \to [n]$ be Hessenberg functions. If for any $i \in [n]$ $h(i) \leq h'(i)$ then we say $h \preceq h'$.

Example



Dual Hessenberg functions

For a Hessenberg function $h : [n] \to [n]$, we can define a new Hessenberg function $h^* : [n] \to [n]$ as follows.

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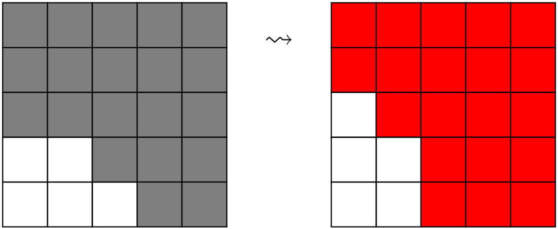

Regular semisimple Hessenberg varieties

Let S be an $n \times n$ regular semisimple matrix (i.e. an $n \times n$ diagonalizable matrix with n distinct eigenvalues) and $h : [n] \to [n]$ be a Hessenberg function, we can define a subvariety of the full flag variety $\mathrm{Flag}(\mathbb{C}^n)$ as follows.

$$\mathrm{Hess}(S, h) := \{V_\bullet \in \mathrm{Flag}(\mathbb{C}^n) | SV_i \subseteq V_{h(i)}\}.$$

$\mathrm{Hess}(S, h)$ is called a **regular semisimple Hessenberg variety**.

Assumption and notation

- The general linear group $\mathrm{GL}_n(\mathbb{C})$ left acts on $\mathsf{Flag}(\mathbb{C}^n)$ so we have $\mathsf{Hess}(gSg^{-1}, h) = g\,\mathsf{Hess}(S, h)$ in $\mathsf{Flag}(\mathbb{C}^n)$ for all $g \in \mathrm{GL}(\mathbb{C}^n)$. It follows that $\mathsf{Hess}(gSg^{-1}, h) \cong \mathsf{Hess}(S, h)$ for all $g \in \mathrm{GL}(\mathbb{C}^n)$. Hence, we can always assume that the regular semisimple matrix S is a diagonal matrix.

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1. If $h(i) = n$ for all $i \in [n]$, then for any S we have $\mathrm{Hess}(S, h) = \mathrm{Flag}(\mathbb{C}^n)$.
2. If $h(i) = i + 1$ for $i = 1, 2, \cdots, n-1$, then $\mathrm{Hess}(S, h)$ is called the permutohedral variety which is the smooth projective toric variety corresponding to the fan consisting of the collection of Weyl chambers in type A_{n-1}.

Properties

The following properties on regular semisimple Hessenberg varieties can be found in De Mari, Procesi and Shayman's nice paper [4].

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1. $\mathrm{Hess}(S, h)$ is smooth;
2. $\dim_{\mathbb{C}} \sum_{i=1}^{n}(h(i) - i)$.

Torus action on Hess(S, h)

Let T be the following complex torus consisting of diagonal matrices:

$$T := \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \;\middle|\; g_i \in \mathbb{C}^* \text{ for all } 1 \leq i \leq n \right\}.$$

Then T acts on Flag($\mathbb{C}^n$) via the $\mathrm{GL}_n(\mathbb{C})$-action on Flag($\mathbb{C}^n$). Since the matrix S defining Hess(S, h) is diagonal, all the elements of T commute with S. Therefore, the T-action on Flag($\mathbb{C}^n$) preserves Hess(S, h), and hence T acts on Hess(S, h).

The fixed points set $\mathrm{Hess}(S,h)^T$

- Recall that we have $\mathrm{Flag}(\mathbb{C}^n)^T = \mathfrak{S}_n$ by identifying $w \in \mathfrak{S}_n$ and $V_\bullet$, where $V_i = \mathbb{C}e_{w(1)} \oplus \mathbb{C}e_{w(2)} \oplus \cdots \oplus \mathbb{C}e_{w(i)}$ for all $1 \le i \le n$. In fact, $\mathrm{Hess}(S,h)$ contains all the T-fixed points of $\mathrm{Flag}(\mathbb{C}^n)^T$:

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$$\mathrm{Hess}(S, h)^T = \mathrm{Flag}(\mathbb{C}^n)^T = \mathfrak{S}_n.$$

Problem If X is a T-invariant nonsingular subvariety of $\mathrm{Flag}(\mathbb{C}^n)$ such that $\chi(X) = \chi(\mathrm{Flag}(\mathbb{C}^n))$, then should X be isomorphism to some $\mathrm{Hess}(S, h)$?

Remark

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1. $H_*(X(h)) \cong H_*(\mathrm{Hess}(S,h))$;
2. $X(h)^{\mathbb{T}} \cong \mathrm{Hess}(S,h)^T$.

Details can be found in [2].

Weights

1. **Weight:** An algebraic homomorphism $\mu : T \to \mathbb{C}^*$. Hence

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3. **Regular dominant weights:**
$$\{\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{Z}^n | \mu_1 > \mu_2 > \cdots > \mu_n\}$$

Line bundles associated to weights

- $\mathbb{C}_\mu$ be the one dimensional complex representation of T with weight $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$. Consider the projection $B \twoheadrightarrow T$, then $\mathbb{C}_\mu$ is also B-module given by

$$\begin{pmatrix} b_1 & * & * & * \\ & b_2 & * & * \\ & & \ddots & * \\ & & & b_n \end{pmatrix} \cdot v = b_1^{\mu_1} b_2^{\mu_2} \cdots b_n^{\mu_n} v.$$

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- $L_\mu := G \times^B \mathbb{C}_\mu^*$, the B-action on $G \times C_\mu^*$ is given by

$$(g, v)b := (gb, b^{-1} \cdot v) = (gb, \mu(b)v).$$

We also denote $L_\mu|_{\mathrm{Hess}(S,h)}$ by L_μ.

Ample and nef line bundles

Proposition ([3])

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Corollary

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The anti-cononical bundle

Proposition

$$\mathcal{O}(-K_{\mathrm{Hess}(S,h)}) \cong L_{\xi_h}$$

where

$$\xi_h = \sum_{1 \leq i < j \leq h(i)} (t_i - t_j) \textit{ and } t_i = (0, 0, \cdots, 1, 0, \cdots, 0)$$

Example

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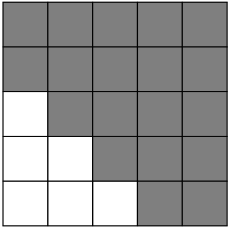
If $h = (2, 3, 4, 5, 5)$ then $\xi_h = t_1 - t_5$.



When $\mathrm{Hess}(S, h)$ is Fano

Fano: If $\mathcal{O}(-K_{\mathrm{Hess}(S,h)})$ is ample.

Theorem (Abe-Fujita-Z)

Assume that $h(i) \geq i + 1$ *for all* $1 \leq i \leq n - 1$*. Then* $\mathrm{Hess}(S, h)$ *is Fano if and only if*

$$h = (k + 1, k + 2, \cdots, n, n, \cdots, n)$$

such that $\frac{n-1}{2} \leq k \leq n - 1$.

Example

Example

1. If $h = (2, 3, 3)$, $\mathrm{Hess}(S, h)$ is Fano.
2. If $h = (2, 3, 4, 4)$, $\mathrm{Hess}(S, h)$ is not Fano.

Definition

Weak Fano: If $\mathcal{O}(-K_{\mathrm{Hess}(S,h)})$ is nef and big.

Proposition ([5])

If $\mathcal{O}(-K_{\mathrm{Hess}(S,h)})$ is nef then the following conditions are equivalent:

1. $\mathcal{O}(-K_{\mathrm{Hess}(S,h)})$ *is big;*
2. $\int_{\mathrm{Hess}(S,h)} c_1(\mathcal{O}(-K_{\mathrm{Hess}(S,h)}))^d > 0$ *where* $d = \dim_{\mathbb{C}} \mathrm{Hess}(S,h)$.

When $\mathrm{Hess}(S, h)$ is weak Fano

Theorem (Abe-Fujita-Z)

Assume that $h(i) \geq i+1$ *for all* $1 \leq i \leq n-1$. *Then* $\mathrm{Hess}(S, h)$ *is weak Fano if and only if*

$$h(i) - h(i+1) + 2 - h^*(n+1-i) + h^*(n-i) \geq 0$$

for all $1 \leq i \leq n-1$.

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