Generalized equivariant cohomologies of GKM orbifolds

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(Joint work with Soumen Sarkar)

24 March, 2022

Toric Topology 2022 in Osaka

GKM manifold

Definition (Guillemin and Zara [1])

Let M be $2n$-dimensional compact manifold with an action of

k-dimensional compact abelian Lie group G. M is said to be

GKM manifold if the following holds

1. $|M^G| < \infty$.

2. M has a G-equivariant almost complex structure.

3. The weights $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of the isotropy

   representations of G on T_pM are pairwise linearly

   independent for all $p \in M^T$.

[1] GUILLEMIN, V; ZARA, C. *1-skeleta, Betti numbers, and equivariant cohomology*. Duke Math. J. 2001, no. 107, 283-349.

GKM graph

Let
$$M_1 := \{x \in M : dim(G(x)) \leq 1\}.$$

Then M_1 has a structure of a n-valent graph $\Gamma = (V, E)$.

Here the set of vertices V is the set of fixed points M^G.

E is the set of edges corresponding to invariant 2-spheres

connecting the fixed points.

To each oriented edge $e \in E$ we assign an weight $\alpha(e)$ of the

isotropy representation of T on T_pM, if $p = i(e)$.

The pair (Γ, α) is called GKM-graph.

We want to calculate the generalized equivariant cohomology

theory (equivariant cohomology, equivariant K- theory

and equivariant cobordism theory) of GKM manifold *M*?

Can we generalize this for GKM orbifolds and to a more general

broader category?

[2] defined the graph equivariant cohomology $H(\Gamma, \alpha)$ of (Γ, α)

and proved that if *M* is equivariantly formal then $H(\Gamma, \alpha)$ is

isomorphic to $H_G(M)$.

[2]GUILLEMIN, V; ZARA, C. *1-skeleta, Betti numbers, and equivariant cohomology*. Duke Math. J. 2001, no. 107, 283-349.

Filtration of a regular graph

Let $\Gamma = (V, E)$ be an n valent graph, where V is the vertices

and E is the edges of Γ.

Let $b_0 \in V$, $V_0 = \{b_0\}$ and $\Gamma_0 := (V_0, E_0)$ where $E_0 = \emptyset$.

Next we consider $b_1 \in V - V_0$ which is adjacent to b_0.

Let $V_1 = \{b_0, b_1\}$ and E_1 be the edge joining b_0 and b_1.

Define $\Gamma_1 := (V_1, E_1)$.

Suppose, inductively, we define $\Gamma_k := (V_k, E_k)$ where

$V_k = \{b_0, b_1, \ldots, b_k\}$ and E_k is the edges in E whose vertices
are in V_k. Let

$k' := \min\{\ell \in \{0, 1, \ldots, k\} \mid b_\ell$ is adjacent to a vertex in $V - V_k\}$.

Filtration of a regular graph

Now we consider $b_{k+1} \in V - V_k$ satisfying that b_{k+1} is adjacent to $b_{k'}$.

Let $V_{k+1} := \{b_0, b_1, \ldots, b_k, b_{k+1}\}$ and E_{k+1} is the edges in E whose vertices are in V_{k+1}. So

$$E_{k+1} := \{e \in E \mid V(e) \subset V_{k+1}\} = E_k \sqcup \{e \in E \mid b_{k+1} \in V(e) \subset V_{k+1}\}.$$

Then define $\Gamma_{k+1} := (V_{k+1}, E_{k+1})$. This process stops when there is no remaining vertices. Therefore

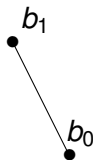$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma \tag{1}$$

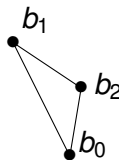gives a filtration of Γ, since Γ is a connected graph, where

$m + 1 = |V|$.
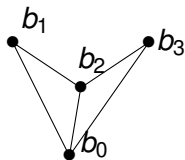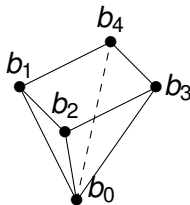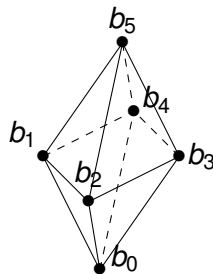
Exponential map

Let
$$B(0_p, r) = \{v \in T_pM \mid ||v|| = R_p(0_p, v) < r\}$$

where 0_p is the zero vector in T_pM.

We have the exponential map at p is the map

$$exp_p \colon B(0_p, r) \to M$$

Then exp_p is G-equivariant and it is a diffeomorphism on an

open neighborhood of 0_p in T_pM to an open neighborhood of p

in M

G-invariant filtration

Consider that map $h\colon M^1 \to \Gamma$.

Consider $p \in M^G$ and $h(p) = b_i \in V$ for some $0 \le i \le m$.
Now
$$T_p M = V(\alpha_1) \oplus V(\alpha_2) \oplus \cdots \oplus V(\alpha_n).$$

Recall the filtration of $\Gamma = (V, E)$ as in (1).

For $i \in \{1, 2, \ldots, m\}$ let $F_i = E_i - E_{i-1}$ and $e_1, e_2, \ldots, e_{d_i}$ be the

edges in F_i containing b_i with weights $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{d_i}^{(i)}$
respectively.

Using exponential map, there exist a *G*-invariant submanifold

M_i which is the homeomorphic image of a *G*-invariant

G-invariant filtration

neighbourhood of the origin in $\oplus_{j=1}^{d_i} V(\alpha_j^{(i)})$.

Then M_i is equivariantly contractible to $h^{-1}(b_i) = p$.

We consider the subset $M_i \subseteq M$ which is maximal with this

property. Let $M_0 = h^{-1}(b_0) \cong \{pt\}$ and

$$X_j = \cup_{i=0}^{j} M_i \subset M$$

for $j = 0, 1, \ldots, m = |V|$.

Note that $h^{-1}(E_j \setminus E_{j-1})$ is the one skeleton of $X_j \setminus X_{j-1}$ for all

$j \in \{1, 2, \ldots, m\}$.

Buildable GKM manifold

Definition

A GKM manifold *X* equipped with the *G*-action is called

build-able if the filtration constructed above will stop at *M*. i.e,

$$\{pt\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = M \tag{2}$$

is a *G*-invariant stratification for some $m \geq 1$.

Example

quasitoric manifolds, Grassmann manifolds are some well

known examples of buildable GKM manifolds.

Theorem (Brahma,Sarkar)

Let M be a buildable GKM manifold and $E_G^* = H_G^*$, K_G^* or MU_G^*.

Then the generalized G-equivariant cohomology of M with

integer coefficients is given by

$$E_G^*(M) = \left\{ (x_j) \in \bigoplus_{j=0}^m E_G^*(b_j) \mid e_G(\xi^{js}) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j \right\}.$$

GKM Orbifold

Let Y be an $2n$ dimensional G-orbifold and $p \in Y^G$ be an isolated fixed point. Then there is an orbifold chart $(\widetilde{U}, \xi, H)$ over a neighborhood $U \subset Y$ of p and a finite covering $\widetilde{G}$ of G such that $\widetilde{G}$ acts on $\widetilde{U}$ effectively, the map $\xi \colon \widetilde{U} \to U$ preserves the respective group actions, $\widetilde{p}$ is the $\widetilde{G}$ fixed point in $\widetilde{U}$ with $\widetilde{p} = \xi^{-1}(p)$ and $\widetilde{G}/H \cong G$. See [GGKRW [3]]

Then the tangent space of $\widetilde{U}$ at $\widetilde{p}$ can be decomposed as

$$T_{\widetilde{p}}\widetilde{U} \cong V(\widetilde{\alpha}_1) \oplus \cdots \oplus V_n(\widetilde{\alpha}_n)$$

[3]Galaz-García, Fernando and Kerin, Martin and Radeschi, Marco and Wiemeler, Michael, *Torus orbifolds, slice-maximal torus actions, and rational ellipticity* Int. Math. Res. Not. IMRN **18**(2018), 5786–5822.

Orbifold GKM graph

Let α_i is the image of $\widetilde{\alpha}_i$ under the Lie algebra map

$$L(\widetilde{G}) \to L(G)$$

induced by the covering homomorphism $\widetilde{G} \to G$.

We say that $\alpha_1, \ldots, \alpha_n$ are the characters of the irreducible

G-representations of $T_p Y$ at p.

Then similarly to the GKM manifold we can define the GKM

orbifold and the corresponding GKM graph Γ of the orbifold.

Vertex set corresponds to the set of all fixed point and there is

an edge between two fixed point if there exist an invariant

spindle connecting those two fixed points.

Let Y be a GKM orbifold and (Γ, α) the corresponding GKM graph.

Now we can define the filtration of the graph $\Gamma = (V, E)$.

Let p be an isolated fixed point such that $h(p) = b_i$,

where $b_i \in V$ (the set of vertices of Γ) with $i \geq 1$.

Let $\{\widetilde{\alpha}_1^{(i)}, \ldots, \widetilde{\alpha}_{d_i}^{(i)}\} \subset \{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n\}$ be the weights

corresponding to the edges in $F_i = E_i - E_{i-1}$.

Then there exists a $\widetilde{G}$-invariant submanifold $\widetilde{M}_i$ of $\widetilde{U}$ containing

$\widetilde{p}$ and $T_{\widetilde{p}}\widetilde{M}_i = \oplus_{j=1}^{d_i} V(\widetilde{\alpha}_j^{(i)})$.

G-invariant filtration

We denote $\xi(\widetilde{M}_i)$ by M_i. Let $H_i := \{h \in H \mid h\widetilde{M}_i = \widetilde{M}_i\}$. Define

$$G_i = H/H_i. \tag{3}$$

Suppose the subset M_i is *G*-equivariantly homeomorphic to

$\mathbb{C}^{d_i}/G_i$ and maximal with this property. Let

$M_0 = h^{-1}(b_0) \cong \{pt\}$ and

$$X_j := \cup_{i=0}^{j} M_i \subset Y$$

for $j = 0, 1, \ldots, m = |V|$. The above observation leads the

following.

Buildable GKM orbifold

Definition

A GKM orbifold Y is called build-able if there is a G-invariant stratification

$$\{pt\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = Y \qquad (4)$$

corresponding to a filtration

$$\{pt\} = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma$$

of its GKM graph such that $h^{-1}(E_j \setminus E_{j-1})$ is the one skeleton of $X_j \setminus X_{j-1}$ and $X_j \setminus X_{j-1}$ is G-equivariantly homeomorphic to $\mathbb{C}^{d_j}/G_j$ for some finite group G_j for $j = 0, 1, 2, \ldots, m$.

In addition, if the group G_j defined in (3) is trivial for any j, then Y is called a 'divisive' GKM orbidold.

Some buildable GKM orbifold

Example

Quasitoric orbifolds, weighted Grassmann orbifolds, toric varieties over almost complex polytope are some examples of buildable GKM orbifold.

Divisive weighted projective spaces, retractable toric orbifolds, divisive toric varieties are some examples of divisive GKM orbifold.

Orbifold vector bundle

Let B be an effective orbifold and $\mathcal{A} := \{(\widetilde{V}_i, G_i, \phi_i) \mid i \in \mathcal{I}\}$ an orbifold atlas on B. Now assume that $(\widetilde{X}_i, \widetilde{V}_i, \widetilde{P}_i)$ is a G_i invariant ℓ-dimensional vector bundle for $i \in \mathcal{I}$ such that if there exists an embedding of orbifold chart $\lambda\colon (\widetilde{V}_i, G_i, \phi_i) \to (\widetilde{V}_j, G_j, \phi_j)$ then $\widetilde{X}_i = \lambda^*(\widetilde{X}_j)$. Let $\pi_i\colon \widetilde{X}_i \to X_i = \frac{\widetilde{X}_i}{G_i}$ be the orbit map.

Then $(\widetilde{X}_i, G_i, \pi_i)$ is an orbifold chart on X_i for all $i \in \mathcal{I}$.

The collection $\{(\widetilde{X}_i, G_i, \pi_i) \mid i \in \mathcal{I}\}$ is an orbifold atlas for X.

This gives an ℓ-dimensional orbifold vector bundle $P\colon X \to B$.

The triple (X, B, P) is said to be an ℓ-dimensional orbifold vector bundle.

Definition

Let X and B be two G-spaces such that the orbifold vector bundle $\widetilde{P}_i \colon \widetilde{X}_i \to \widetilde{V}_i$ is G-vector bundle and the action of G (on $\widetilde{V}_i$ and $\widetilde{X}_i$) commutes with the action of G_i for all $i \in \mathcal{I}$. Then the map $P \colon X \to B$ constructed above is called an orbifold G-vector bundle.

Proposition (Thom isomorphism for orbifold *G*-vector bundle)

Let E_G^* be one of H_G^* and K_G^*, and $P \colon X \to B$ an ℓ-dimensional orbifold G-vector bundle as in Definition 7. Suppose that G- and G_i-representations commute on each fiber of $\widetilde{P}_i \colon \widetilde{X}_i \to \widetilde{V}_i$ for each $i \in \mathcal{I}$. If B is compact, then the map

$$P^* \colon E_G^*(B; \mathbb{Q}) \to E_G^{*+\ell}(X, X_0; \mathbb{Q})$$

is an isomorphism.

Equivariant stratification

Now we consider the following *G*-invariant stratification

$$\{pt\} = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m \tag{5}$$

of a *G*-space *Y* such that $Y = \bigcup_{j=0}^{m} Y_j$ and Y_j/Y_{j-1} is homeomorphic to the Thom space $Th(X_j)$ of an orbifold *G*-vector bundle $\xi^j \colon X_j \to B_j$.

Therefore *Y* can be built from Y_0 inductively by attaching **q**-disc bundles $D(X_j)$ to Y_{j-1} via some *G*-equivariant map

$$\eta_j \colon S(X_j) \to Y_{j-1},$$

for $j = 1, \ldots, m$. This gives the following cofibration

$$Y_{j-1} \to Y_j \to Th(X_j).$$

Equivariant cohomology theory

Let Y be a G-space with the G-stratification as in (5) which satisfies the following assumptions.

(A1) Each orbifold G-vector bundle $\xi^j \colon X_j \to B_j$ is E-orientable and has the following decomposition

$$(\xi^j \colon X_j \to B_j) \cong \bigoplus_{s<j}(\xi^{js} \colon X_{js} \to B_j)$$

into E-orientable orbifold G-vector bundles ξ^{js}, (where X_{js} can be trivial).

(A2) The restriction of the attaching map $\eta_j \colon S(X_j) \to Y_{j-1}$ on $S(X_{js})$ satisfies

$$\eta_j|_{S(X_{js})} = f_{js} \circ \xi^{js}$$

for some G-equivariant map $f_{js} \colon B_j \to B_s \subset Y_{j-1}$, for $s < j$.

(A3) The equivariant Euler classes $\{e_G(\xi^{js}); s < j\}$ are not zero divisors and pairwise relatively prime in $E_G^*(B_j)$.

Equivariant cohomology theory

Proposition

Let Y be a G-space with a G-stratification as in (5) *such that assumptions* (A1),(A2) *and* (A3) *are satisfied. Then the equivariant cohomology $E_G^*(Y)$ of Y is given by*

$$E_G^*(Y) = \left\{ (x_j) \in \bigoplus_{j=0}^{m} E_G^*(B_j) \mid e_G(\xi^{js}) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j \right\}.$$

If all the G_i's are trivial then the above proposition also holds for MU_G^*, [4]

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[4]Harada, Megumi and Henriques, André and Holm, Tara S, *Computation of generalized equivariant cohomologies of Kac-Moody flag varieties* Adv. Math. **197**(2005), no.1, 198-221.

Equivarint cohomology theory of GKM orbifold

Proposition (Brahma,Sarkar)
If Y is a build-able GKM orbifold with filtration as in (4)*, then it*

satisfies the conditions (A1), (A2) and (A3).

Theorem (Brahma,Sarkar)
Let Y be a build-able GKM orbifold with the filtration as in (4)

and $E_G^* = H_G^*$ or K_G^*. Then the generalized G-equivariant

cohomology of Y is given by

$$E_G^*(Y; \mathbb{Q}) = \left\{ (x_j) \in \bigoplus_{j=0}^{m} E_G^*(b_j) \mid e_G(\xi^{js}) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j \right\}$$

Equivariant cohomology theory of Divisive GKM orbifold

Theorem (Brahma,Sarkar)

Let Y be a divisive simplicial GKM orbifold complex and

$E_G^* = H_G^*$, K_G^* *or* MU_G^*. *Then the generalized G-equivariant*

cohomology of Y with integer coefficients is given by

$$E_G^*(Y; \mathbb{Z}) = \Big\{ (x_j) \in \bigoplus_{j=0}^{m} E_G^*(b_j; \mathbb{Z}) \mid e_G(\xi^{js}) \text{ divides } x_j - f_{js}^*(x_s)$$
$$\text{for all } s < j \Big\}.$$

Arigato/**ThankYou**