Constructing Bieberbach Groups from a quotient group of the orbit braid group

Shuya Cai @ Fudan University

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Charlap L. S., Bieberbach Groups and Flat Manifolds;
Dekimpe K., Almost-Bieberbach groups: Affine and Polynomial Structures.

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Definition

A discrete and uniform subgroup Π of $\mathbb{R}^n \rtimes O(n, \mathbb{R}) \subset \mathrm{Aff}(\mathbb{R}^n)$ is said to be a crystallographic Group of dimension n. If in addition Π is torsion free then Π is called a Bieberbach group of dimension n.

If Φ is a group, an integral representation of rank m of Φ is defined to be a homomorphism $\Theta : \Phi \to Aut(\mathbb{Z}^m)$. Two such representations are said to be equivalent if their images are conjugate in $Aut(\mathbb{Z}^m)$. We say that Θ is a faithful representation if it is injective.

Lemma

Let Π be a group. Then Π is a crystallographic group if and only if there exists an integer $n \in \mathbb{N}$ and a short exact sequence

$$1 \to \mathbb{Z}^n \to \Pi \xrightarrow{\zeta} \Phi \to 1$$

such that:

(a) *Φ is finite, and*

(b) *the integral representation $\Theta : \Phi \to Aut(\mathbb{Z}^n)$, induced by conjugation on $\mathbb{Z}$ and defined by $\Theta(\varphi)(x) = \pi x \pi^{-1}$, for all $x \in \mathbb{Z}^n, \varphi \in \Phi$, where $\pi \in \Pi$ is such that $\zeta(\pi) = \varphi$, is faithful.*

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The integer n is called the dimension of Π. The finite group Φ is called the holonomy group of Π. And the integral representation $\Theta : \Phi \to Aut(\mathbb{Z}^n)$ is called the holonomy representation of Π.

$$1 \to \mathbb{Z}^n \to \Pi \xrightarrow{\zeta} \Phi \to 1$$

Corollary

Let Π be a crystallographic group of dimension n and holonomy group Φ, and let H be a subgroup of Φ. $\zeta^{-1}(H)$ is a crystallographic subgroup of Π of dimension n with holonomy group H.

Definition

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Theorem (the first Bieberbach Theorem)

There is a correspondence between Bieberbach groups and fundamental groups of closed flat Riemannian manifolds.

Theorem (Wolf J.A.)

The holonomy group of the corresponding flat manifold M is isomorphic to the group Φ.

Theorem (Auslander and Kuranishi)

Any finite group is the holonomy group of some flat manifold.

Lemma

Let M be the flat manifold whose fundamental group is the Bieberbach group Π. Then M is orientable if and only if the integral representation $\Theta : \Phi \to \mathrm{Aut}(\mathbb{Z}^n)$ satisfies

$$\mathrm{Im}(\Theta) \subset \mathrm{SO}(n, \mathbb{Z}).$$

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Lemma

Let M be the flat manifold whose fundamental group is the Bieberbach group Π. The holonomy group Φ is generated by $U_1, \cdots, U_s$. Then the first Betti number of M is:

$$\begin{aligned} b_1(M) &= \mathrm{rankH_1(M, \mathbb{Z})} = \mathrm{rank}\frac{\pi}{[\pi, \pi]} \\ &= n - \mathrm{rank}(\Theta(\mathrm{U_1}) - \mathrm{I}, \cdots, \Theta(\mathrm{U_s}) - \mathrm{I}). \end{aligned}$$

Lemma

Let M be the flat manifold whose fundamental group is the Bieberbach group Π. The holonomy group Φ is cyclic. Suppose that $A = \Theta(1)$. Then M supports an Anosv diffeomorphism if and only if A has none of the following numbers as simple eigenvalues:

$$\pm 1, \pm i, \pm \omega, \pm \omega^2, where \quad \omega^3 = 1.$$

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Lemma

Let M be the flat manifold whose fundamental group is the Bieberbach group Π. Then M is Kähler if and only if

a. *the dimension of M is even, and*
b. *each $\mathbb{R}$-irreducible summand of φ which is also $\mathbb{C}$-irreducible occurs with an even multiplicity.*

Definition

The Artin braid group B_n on n strands is defined by the presentation:

- *generators:* $\sigma_1, \cdots, \sigma_{n-1}$;
- *relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad i = 1, \cdots, n-2.$$

Definition

The Artin pure braid group P_n is defined by the presentation:

- *generators:* $S_{ij}, 1 \leq i < j \leq n$;
- *relations:*

$$S_{rs}^{-1}S_{ij}S_{rs} = \begin{cases} S_{ij}, & r < s < i < j; \\ S_{ij}, & i < r < s < j; \\ S_{rj}S_{ij}S_{rj}^{-1}, & r < i = s < j; \\ (S_{ij}S_{sj})S_{ij}(S_{ij}S_{sj})^{-1}, & r = i < s < j; \\ (S_{rj}S_{sj}S_{rj}^{-1}S_{sj}^{-1})S_{ij}(S_{rj}S_{sj}S_{rj}^{-1}S_{sj}^{-1})^{-1}, & r < i < s < j. \end{cases}$$

Proposition (Gonçalves, Guaschi, Ocampo)

Let $n \geqslant 2$. There is a short exact sequence:

$$1 \to \mathbb{Z}^{\frac{n(n-1)}{2}} \to B_n/[P_n, P_n] \xrightarrow{\bar{s}} \Sigma_n \to 1,$$

and the middle group $B_n/[P_n, P_n]$ is a crystallographic group.

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Corollary

Let $n \geqslant 3$, and let H be a subgroup of Σ_n. Then the group $\widetilde{H}_n$ defined by

$$\widetilde{H}_n = s^{-1}(H)/[P_n, P_n] \tag{1}$$

is a crystallographic group of dimension $\frac{n(n-1)}{2}$ with holonomy group H.

Theorem (Gonçalves, Guaschi, Ocampo)

If $n \geqslant 3$ then the quotient group $B_n/[P_n, P_n]$ has no finite-order element of even order.

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Corollary

Let $n \geqslant 3$, and let H be a 2-subgroup of Σ_n. Then the group $\widetilde{H}_n$ given by equation (1) is a Bieberbach group of dimension $\frac{n(n-1)}{2}$.

Consider the cyclic subgroup $H_{2^d,k} \subset \Sigma_n$:

$$H_{2^d,k} = \langle \mu \rangle,$$
$$\mu = (2^d, \cdots, 1)(2 \cdot 2^d, \cdots, 1 + 2^d) \cdots (k, \cdots, k - (2^d - 1))$$

with $k = 2^d m$ such that $2^d \leqslant k \leqslant n$ and m is a positive integer. Then $\widetilde{H}_{2^d,k} = s^{-1}(H_{2^d,k})/[P_n, P_n]$ is a Bieberbach group with

holonomy group $H_{2^d,k}$ and holonomy representation

$$\psi_{H_{2^d,k}} : H_{2^d,k} \to \mathrm{Aut}(\mathrm{P_n}/[\mathrm{P_n}, \mathrm{P_n}]).$$

Theorem (Ocampo, Rodriguez-Nieto)

Let $k = 2^d m$. Let $\chi_{H_{2^d,k}}$ be the flat manifold of dimension $\frac{n(n-1)}{2}$ with fundamental group $\widetilde{H}_{2^d,k}$ and holonomy group $H_{2^d,k} = \mathbb{Z}_{2^d}$. Then

a. *$\chi_{H_{2^d,k}}$ is orientable if and only if one of n or m is even.*
b. *The first homology group of the flat manifold $\chi_{H_{2^d,k}}$ is $H_1(\chi_{H_{2^d,k}}, \mathbb{Z}) = \mathbb{Z}^{|\mathcal{T}|} \oplus \mathbb{Z}_{2^{d-1}}$, where $\mathcal{T}$ is the transversal of the action by conjugation of $H_{2^d,k}$ on the basis $\{A_{i,j} | 1 \leqslant i < j \leqslant n\}$ of $P_n/[P_n, P_n]$ satisying*

$$|\mathcal{T}| = \frac{k}{2^d} + \frac{k(2n-k-2)}{2^{d+1}} + \frac{(n-k)(n-k-1)}{2}.$$

So, the first Betti number of the flat manifold $\chi_{H_{2^d,k}}$ is

$$b_1(\chi_{H_{2^d,k}}) = \frac{(2^d-1)k^2 - 2kn(2^d-1) + 2^d n^2 - 2^d n}{2^{d+1}}.$$

Theorem (Ocampo, Rodriguez-Nieto)

- *The flat manifold $\chi_{H_{2^d,k}}$ with fundamental group $\widetilde{H}_{2^d,k}$ admits Anosov diffeomorphism if and only if*
  1. $n \geqslant 4$ *in the case* $d = 1$,
  2. $n \geqslant 5$ *in the case* $d = 2$,
  3. $n \geqslant 2^d$ *in the case* $d \geqslant 3$.
- *Suppose that $\frac{n(n-1)}{2}$ is even, so $n = 4q$ or $n = 4q + 1$, for some q. Let $d = 1$, then the flat manifold $\chi_{H_{2^d,k}}$ is Kähler if and only if $n = 4q$ and m is even. Let $d2$ then the flat manifold $\chi_{H_{2^d,k}}$ is Kähler if and only if one of the following conditions holds*
  1. $n = 4q$
  2. $n = 4q + 1$ *and* m *is even.*

Theorem (Ocampo)

Let G be a finite abelian group.

(a) *There exists n and a Bieberbach subgroup Γ_G of $B_n/[P_n, P_n]$ of dimension $\frac{n(n-1)}{2}$ with holonomy group G.*

(b) *The finite abelian group G is the holonomy group of a flat manifold χ_{Γ_G} of dimension $\frac{n(n-1)}{2}$, where n is an integer for which G embeds in the symmetric group Σ_n, and the fundamental group of χ_{Γ_G} is isomorphic to a subgroup Γ_G of $B_n/[P_n, P_n]$.*

Theorem (Ocampo)

Let $q = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ be an odd number, where p_i are distinct odd primes and $r_i \geqslant 1$ for all $1 \leqslant i \leqslant t$. Let χ_{Γ_q} be the flat manifold of dimension $\frac{n(n-1)}{2}$ with fundamental group $\Gamma_q \subset B_n/[P_n, P_n]$ and holonomy group $\mathbb{Z}_q$. Then

a. *χ_{Γ_q} is orientable.*
b. *The first Betti number of χ_{Γ_q} is $b_1(\chi_{\Gamma_q}) = \sum_{i=1}^{t} \frac{p_i^{r_i}-1}{2} + \frac{t(t-1)}{2}$.*
c. *The flat manifold χ_{Γ_q} with fundamental group Γ_q admits Anosov diffeomorphism if and only if $q \neq 3$.*

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- (a) *χ_{Γ_q} is orientable.*
- (b) *The first Betti number of χ_{Γ_q} is $b_1(\chi_{\Gamma_q}) = \sum_{i=1}^{t} \frac{p_i^{r_i}-1}{2} + \frac{t(t-1)}{2}$.*
- (c) *The flat manifold χ_{Γ_q} with fundamental group Γ_q admits Anosov diffeomorphism if and only if $q \neq 3$.*

Theorem (Ocampo)

Let p be an odd prime and let $r \geqslant 1$. Let $\chi_{\Gamma_{p^r}}$ be the flat manifold of dimension $\frac{p^r(p^r-1)}{2}$ with fundamental group $\Gamma_{p^r} \subset B_n/[P_n, P_n]$ and holonomy group $\mathbb{Z}_{p^r}$. Then the flat manifold $\chi_{\Gamma_{p^r}}$ is Kähler if and only if there is an integer $u \geqslant 1$ such that $p^r = 4u + 1$.

Surface braid group

Proposition (Goncalves, Guaschi, Ocampo, Pereiro)

Let M be an orientable, compact connected surface of genus $g \geqslant 1$ without boundary, and let $n \geqslant 2$. Then there exists a split extension of the form:

$$1 \to \mathbb{Z}^{2ng} \to B_n(M)/[P_n(M), P_n(M)] \xrightarrow{\bar{s}} \Sigma_n \to 1.$$

The quotient $B_n(M)/[P_n(M), P_n(M)]$ is a crystallographic group of dimension $2ng$, whose holonomy group is Σ_n.

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Proposition (Goncalves, Guaschi, Ocampo, Pereiro)

Let $M = \mathbb{S}^2$ or N_g, where $g \geqslant 1$. Then for all $n \geqslant 1$, the quotient $B_n(M)/[P_n(M), P_n(M)]$ is not a crystallographic group.

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Generators: $\sigma_1, \cdots, \sigma_{n-1}, a_{i,r}, 1 \leqslant i \leqslant n, 1 \leqslant r \leqslant 2g$.

Relations:

1. *the Artin relations.*
2. $\sigma_i^2 = 1$, *for all* $i = 1, \cdots, n-1$
3. $[a_{i,r}, a_{j,s}] = 1$, *for all* $i, j = 1, \cdots, n$ *and* $r, s = 1, \cdots, 2g$.
4. $\sigma_i a_{j,r} \sigma_i^{-1} = a_{\tau_i(j),r}$ *for all* $1 \leqslant i \leqslant n-1$, $1 \leqslant j$ *and* $1 \leqslant r \leqslant 2g$.

Surface braid group

Theorem (Goncalves, Guaschi, Ocampo, Pereiro)

Let $n \geqslant 2$, and let M be an orientable surface of genus $g \geqslant 1$ without boundary. Let G_n be the cyclic subgroup $\langle (n, n-1, \cdots, 2, 1) \rangle$ of Σ_n. Then there exists a subgroup $\tilde{G}_{n,g}$ of $\sigma^{-1}(G_n)/[P_n(M), P_n(M)] \subset B_n(M)/[P_n(M), P_n(M)]$ that is a Bieberbach group of dimension $2ng$ whose holonomy group is G_n. Further, the centre $Z(\tilde{G}_{n,g})$ of $\tilde{G}_{n,g}$ is a free Abelian group of rank $2g$.

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Theorem (Goncalves, Guaschi, Ocampo, Pereiro)

Let $n \geqslant 2$, and let $\chi_{n,g}$ be a $2ng$-dimension flat manifold whose fundamental group is the Bieberbach group $\tilde{G}_{n,g}$. Then $\chi_{n,g}$ is an orientable Kahler manifold with first Betti number $2g$ that admits Anosov diffeomorphisms.

complex braid groups

Theorem (Marin I.)

For every complex reflection group W, the group $B/[P,P]$ is crystallographic with holonomy group $W/Z(W)$ of dimension $N = |\mathcal{A}|$. The kernel of the projection map $B/[P,P] \to W/Z(W)$ is the subgroup P_0 generated by P^{ab} and $Z_0(B)$. We have $P_0 \cong \mathbb{Z}^N$.

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Theorem (Marin I.)

For every complex reflection group W, the group $B/[P,P]$ has no element of order 2.

Orbit Braid Group

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$G \curvearrowright M$

Definition (orbit configuration space, Xicontencatle)

$$F_G(M,n) = \{(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in M^n | G(\mathbf{x}_i) \bigcap G(\mathbf{x}_j) = \emptyset \quad if \quad i \neq j\}$$

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Definition (Hao Li, Zhi Lü, Fengling Li) (orbit braid group)

$$B_n^{orb}(M,G) \cong \pi_1^E(F_G(M,n), \mathbf{x}, \mathbf{x}^{orb})$$
$$P_n^{orb}(M,G) \cong \pi_1^E(F_G(M,n), \mathbf{x}, G^n(\mathbf{x}))$$
$$B_n(M,G) \cong \pi_1^E(F_G(M,n), \mathbf{x}, \Sigma_n \mathbf{x})$$
$$P_n(M,G) \cong \pi_1(F_G(M,n), \mathbf{x})$$

$\mathbb{Z}_2 \curvearrowright \mathbb{C}$

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Theorem (Hao Li, Zhi Lü, Fengling Li)

$B_n^{orb}(\mathbb{C}, \mathbb{Z}_2)$ *admits the following presentation:*

- *generators:* $\sigma; \sigma_i, i = 1, \cdots, n-1.$
- *relations:*
  1. $\sigma^2 = 1;$
  2. $\sigma\sigma_1\sigma\sigma_1 = \sigma_1\sigma\sigma_1\sigma;$
  3. $\sigma\sigma_i = \sigma_i\sigma, i = 2, \cdots, n-1;$
  4. $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1;$
  5. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$

$\mathbb{Z}_2 \curvearrowright \mathbb{C}$

Theorem

$P_n^{orb}(\mathbb{C}, \mathbb{Z}_2)$ admits the following presentation:

- *generators:* $z_k, k = 1, \cdots, n; S_{ij}, 1 \leq i < j \leq n.$
- *relations:*
  1. $z_k^2 = 1, k = 1, \cdots, n;$
  2. $$z_k S_{ij} z_k = \begin{cases} S_{i,j}, & k < i < j \quad or \quad i < j < k; \\ z_j S_{ij} z_j, & k = i < j; \\ z_j S_{kj} z_j S_{kj}^{-1} S_{ij} S_{kj} z_j S_{kj}^{-1} z_j, & i < k < j; \end{cases}$$
  3. $z_k z_s z_k = z_s S_{ks} z_s S_{ks}^{-1} z_s^{-1}, k < s;$
  4. $$S_{rs}^{-1} S_{ij} S_{rs} = \begin{cases} S_{ij}, & r < s < i < j; \\ S_{ij}, & i < r < s < j; \\ S_{rj} S_{ij} S_{rj}^{-1}, & r < i = s < j; \\ (S_{ij} S_{sj}) S_{ij} (S_{ij} S_{sj})^{-1}, & r = i < s < j; \\ (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1}) S_{ij} (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1})^{-1}, & r < i < s < j. \end{cases}$$

Denote

$$P_n^{orb} = P_n^{orb}(\mathbb{C}, \mathbb{Z}_2); \quad P_n = P_n(\mathbb{C}, \mathbb{Z}_2);$$
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Proposition

$B_n^{orb}/[P_n^{orb}, P_n^{orb}]$ *is not a crystallographic group.*

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We obtain

$$1 \to P_n/[P_n, P_n] \to B_n^{orb}/[P_n, P_n] \xrightarrow{\bar{s}} (\mathbb{Z}_2)^n \rtimes \Sigma_n \to 1.$$

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$$1 \to P_n/[P_n, P_n] \to B_n^{orb}/[P_n, P_n] \xrightarrow{\bar{s}} (\mathbb{Z}_2)^n \rtimes \Sigma_n \to 1.$$

Remark

$\Theta : \mathbb{Z}_2^n \rtimes \Sigma_n \to Aut(\mathbb{Z}^{n(n-1)})$ *induced by the conjugation is not faithful.*

Lemma

Let $E = s^{-1}((-1, \cdots, -1), 1)$. *Then* $(E \bigcup P_n)/[P_n, P_n] \cong \mathbb{Z}^{n(n-1)}$ *is a normal subgroup of* $B_n^{orb}/[P_n, P_n]$ *and it is generated by:*

$$\omega = \sigma(\sigma_1 \sigma \sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_1 \sigma \sigma_1 \cdots \sigma_{n-1}),$$
$$S_{n-1,n}, S_{i,j}, 1 \leqslant |i| < n-1, 1 < j \leqslant n, |i| < j.$$

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$$\omega = \sigma(\sigma_1\sigma\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1\sigma\sigma_1\cdots\sigma_{n-1}),$$
$$S_{n-1,n}, S_{i,j}, 1 \leqslant |i| < n-1, 1 < j \leqslant n, |i| < j.$$

Proposition

Let $n \geqslant 3$ *and* $F = Z((\mathbb{Z}_2)^n \rtimes \Sigma_n) = \langle((-1, \cdots, -1), 1)\rangle$. *Then there is a short exact sequence:*

$$1 \to \mathbb{Z}^{n(n-1)} \to B_n^{orb}/[P_n, P_n] \xrightarrow{\bar{s}} (\mathbb{Z}_2)^n \rtimes \Sigma_n/F \to 1,$$

and $B_n^{orb}/[P_n, P_n]$ *is a crystallographic group.*

Theorem

Let $n \geqslant 3$. And let $\vartheta \in \mathbb{Z}_2^n \rtimes \Sigma_n/F$ satisfy the following conditions:

- *ϑ is of order 2*
- *ϑ is not conjugate to $((-1, 1, \cdots, 1), 1)$.*

Then for $H = <\vartheta>$, $\widetilde{H} = \sigma^{-1}(H)/[P_n, P_n]$ is a Bieberbach group of dimension $n(n-1)$ with the holonomy group H.

Theorem

Let $n \geqslant 3$*.And let* $\vartheta \in \mathbb{Z}_2^n \rtimes \Sigma_n/F$ *satisfy the following conditions:*

- ϑ *is of order* 2
- ϑ *is not conjugate to* $((-1, 1, \cdots, 1), 1)$.

Then for $H =< \vartheta >$, $\widetilde{H} = \sigma^{-1}(H)/[P_n, P_n]$ *is a Bieberbach group of dimension* $n(n-1)$ *with the holonomy group* H.

Corollary

$B_n^{orb}/[P_n, P_n]$ *has no elements of order* $2k, k \geqslant 2$.

Theorem

Let $n \geqslant 3$.And let $\vartheta \in \mathbb{Z}_2^n \rtimes \Sigma_n/F$ satisfy the following conditions:

- *ϑ is of order 2*
- *ϑ is not conjugate to $((-1, 1, \cdots, 1), 1)$.*

Then for $H =< \vartheta >$, $\widetilde{H} = \sigma^{-1}(H)/[P_n, P_n]$ is a Bieberbach group of dimension $n(n-1)$ with the holonomy group H.

Corollary

$B_n^{orb}/[P_n, P_n]$ has no elements of order $2k, k \geqslant 2$.

Theorem

Let $n \geqslant 3$ and let m be odd integers greater than or equal to 3. Then $B_n^{orb}/[P_n, P_n]$ possesses infinitely many elements of order m.

Consider the cyclic subgroup $G \subset (\mathbb{Z}_2)^n \rtimes \Sigma_n/F$:

$$G = \langle \mu = ((\varepsilon_1, \cdots, \varepsilon_n), \theta) \rangle,$$
$$\theta = (2^d \cdots 1) \cdots (m2^d \cdots (m-1)2^d + 1),$$

where $d \geqslant 1, 0 \leqslant m2^d \leqslant n$.

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$$1 \to \mathbb{Z}^{(2^r+1)(2^r+2)} \to B_{2^r+1}^{orb}/[P_{2^r+1}, P_{2^r+1}] \xrightarrow{\bar{s}} (\mathbb{Z}_2)^{2^r+1} \rtimes \Sigma_{2^r+1} \to 1.$$

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$$\overline{\omega}_i = ((1, 1, \cdots, 1, -1, 1, \cdots, 1), 1), \quad 1 \leqslant i \leqslant 2^r,$$
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Let $F = \langle((-1, \cdots, -1), 1)\rangle \subset (\mathbb{Z}_2)^{2^r} \rtimes \mathbb{Z}_{2^r}$. $(\mathbb{Z}_2)^{2^r} \rtimes \mathbb{Z}_{2^r}/F$ is non Abelian.

The subgroup $\Gamma \subset B_{2^r+1}^{orb}/[P_{2^r+1}, P_{2^r+1}]$ is generated by $X_1 \bigcup X_2$, where

$$X_1 = \{\omega_i, 1 \leqslant i \leqslant 2^r, \alpha = \sigma_{2^r} \cdots \sigma_2\},$$
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Γ is a Bieberbach group. The holonomy group is the non Abelian group $(\mathbb{Z}_2)^{2^r} \rtimes \mathbb{Z}_{2^r}/F$.

Definition

An almost-crystallographic group is a discrete subgroup Π *of the semi-direct product* $N \rtimes C$ *that acts properly and discontinuously on* N *such that* N/Π *is compact. If in addition* Π *is torsion free then* Π *is called an almost-Bieberbach group.*

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Infra-nilmanifolds are determined completely by their fundamental groups that are almost-Bieberbach groups.

Theorem (Gasior, Petrosyan, Szczepa)

Let M be an almost-flat manifold with holonomy group F. Then M is orientable if and only if $det = 1$. Suppose M is orientable and a 2-Sylow subgroup of F is cyclic, i.e. $C_{2^t} = \langle q \mid q^{2^t} = 1 \rangle$ for some $t \geqslant 0$. Let π_{ab} denote the abelianisation of the fundamental group π of M.

1. *If $\frac{1}{2}(n - Trace[\theta(q)^{2^{t-1}}]) \not\equiv 2(mod\ 4)$, then M has a Spin structure.*
2. *If $\frac{1}{2}(n - Trace[\theta(q)^{2^{t-1}}]) \equiv 2(mod\ 4)$, then M has a Spin structure if and only if the epimorphism $q_* : \pi_{ab} \to C_{2^t}$ resulting from projection $q : \pi \to C_{2^t}$ factors through a cyclic group of order 2^{t+1}.*

Theorem (Gonalves, Guaschi, Ocampo)

Let $n, k \geqslant 3$. $B_n/\Gamma_k(P_n)$ is an almost-crystallographic group.

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Theorem

$B_n^{orb}/\Gamma_k(P_n)$ *is an almost-crystallographic group.*

Thank you!