Equivariant Bordism of 2-torus Manifolds and Unitary Toric Manifolds

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Outline

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Notations and Definitions

- All the group actions are assumed to be effective here.
- *2-torus manifold* M^n: a smooth closed n-dimensional manifold equipped with an effective smooth $\mathbb{Z}_2^n$-action.
- *unitary toric manifold* M^{2n}: an oriented closed smooth manifold M^{2n} with an effective T^n-action such that its tangent bundle TM admits a T^n-equivariant stable complex structure, and the fixed point set M^{T^n} is nonempty
- A (oriented) G-manifold M is said to be G-*equivariant bord* if there is a compact (oriented) G-manifold W, such that ∂W is invariant under the G-action and $\partial W \cong M$ as (oriented) G-manifolds.
- Two (oriented) G-manifolds M and N are G-*equivariant bordant* if $M \bigcup N$ $(M \bigcup -N)$ is G-equivariant bords.

Equivarint Bordism Classification

- G: a compact Lie group
- $\Omega_n(G)$: the set of (unoriented, orientable, unitary, etc.) G-equivariant bordism classes of n-dimensional G-manifolds.
- $\Omega_n(G)$ forms an abelian group under the disjoint union.
- $\Omega_*(G) = \sum_{n=0}^{\infty} \Omega_n(G)$ forms a graded ring under the cartesian product of manifolds and diagonal G-action.

Quite Hard Problem

Determine $\Omega_n(G)$ and $\Omega_*(G)$.

Equivariant bordisms of 2-torus and unitary toric manifolds

- $\mathcal{Z}_n(\mathbb{Z}_2^n)$: the equivariant bordism groups of 2-torus manifolds
- $\mathcal{Z}_n^U(T^n)$: the equivariant bordism groups of unitary toric manifolds
- $X(\mathbb{Z}_2^n)$ and $X(\mathbb{Z}^n)$: the universal complex of $\mathbb{Z}_2^n$ and $\mathbb{Z}^n$, resp..

In this talk, it'll be shown that

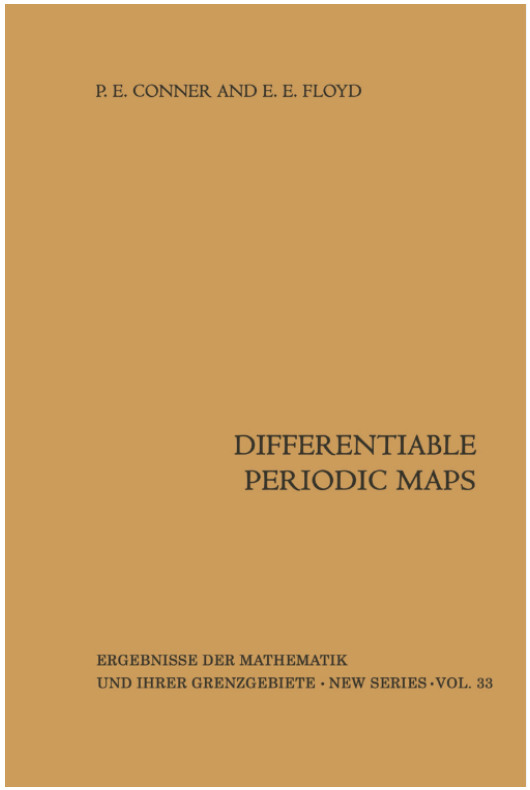
Our Result

$$\mathcal{Z}_n(\mathbb{Z}_2^n) \cong H_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2)$$

$$\mathcal{Z}_n^U(T^n) \cong H_{n-1}(X(\mathbb{Z}^n), \mathbb{Z})$$

Outline

Begin: Work of Conner & Floyd



1964

- $\mathcal{Z}_m(\mathbb{Z}_2^n)$: equivariant bordism group of m-dim $\mathbb{Z}_2^n$-manifolds with isolated fixed points.
- $\mathbb{Z}_2^n$-equivariant bordism class is determined by its fixed point set and their normal bundles.
- $\mathcal{Z}_*(\mathbb{Z}_2) \cong \mathbb{Z}_2$
- $\mathcal{Z}_*(\mathbb{Z}_2^2) \cong \mathbb{Z}_2[u]$, where $u = [(\mathbb{R}P^2, \mathbb{Z}_2^2)]$ and the action of $\mathbb{Z}_2^2$ on $\mathbb{R}P^2$ is standard.

Continued: Conner-Floyd Algebra

- $\mathcal{R}_m(\mathbb{Z}_2^n) = \mathrm{Span}_{\mathbb{Z}_2}\{\text{iso. classes of } m\text{-dim } \mathbb{Z}_2^n\text{-representations}\}$
- $\mathcal{R}_*(\mathbb{Z}_2^n) = \sum_{m=0}^{\infty} \mathcal{R}_m(\mathbb{Z}_2^n)$
- $\mathcal{R}_*(\mathbb{Z}_2^n)$ is a graded algebra over $\mathbb{Z}_2$: the multiplication is the direct sum of representations.
- $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$: set of irreducible $\mathbb{Z}_2^n$-representations.
- $\mathcal{R}_*(\mathbb{Z}_2^n) = \mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \setminus 0]$, polynomial ring whose generators are non-trival irreducible $\mathbb{Z}_2^n$-representations.

Work of Stong

EQUIVARIANT BORDISM AND $(Z_2)^k$ ACTIONS

By R. E. Stong

Duke Math. 1970

- If a $\mathbb{Z}_2^n$-manifold has no fixed point, then it equivariant bords.

key lemma

$$\begin{aligned} \phi_* : & \mathcal{Z}_*(\mathbb{Z}_2^n) \to \mathcal{R}_*(\mathbb{Z}_2^n), \\ & [M] \mapsto \sum_{p \in M^{\mathbb{Z}_2^n}} [\tau_p M] \end{aligned} \tag{1}$$

is a monomorphism between graded algebras.

Work of Lü & Tan

Small Covers and the Equivariant Bordism Classification of 2-torus Manifolds

Zhi Lü and Qiangbo Tan

International Math. Research Notices, 2014

- Consider $\mathbb{Z}_2^n$ as the dual space of $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$
- Instead of the Conner-Floyd algebra $\mathcal{R}_*(\mathbb{Z}_2^n)$, consider the dual polynomial ring $\mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0]$
- For any monomial $s_1 \cdots s_k \in \mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0]$, define

$$d_k(s_1 \ldots s_k) = \begin{cases} \sum_{i=1}^k s_1 \cdots \hat{s}_i \cdots s_k & \text{if } k > 1 \\ 1 & \text{if } k = 1, 0 \end{cases} \tag{2}$$

- $d^2 = 0$, i.e., $(\mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0], d)$ is a chain complex.

Continued

- For a faithful n-dim $\mathbb{Z}_2^n$-rep. $\tau = \rho_1 \cdots \rho_n$
- $\{\rho_1, \cdots, \rho_n\}$ is a basis of $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.
- $\{\rho_1^*, \cdots, \rho_n^*\}$: the corresponding dual basis.
- $\tau^* := \rho_1^* \cdots \rho_n^* \in \mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0]$
- $\mathcal{E}_n := \mathrm{Span}\{\tau \mid \tau \text{ faithful } n\text{-dim rep. }\} \subseteq \mathcal{R}_n(\mathbb{Z}_2^n)$
- $\mathrm{Im}\phi_n \subseteq \mathcal{E}_n$.
- $\mathcal{E}_n^* := \mathrm{Span}\{\tau^* \mid \tau \text{ faithful } n\text{-dim rep. }\} \subseteq \mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0]$.

Lü & Tan

Let $f = \sum \tau \in \mathcal{E}_n$, $f^* := \sum \tau^*$. Then

$$f \in \mathrm{Im}\phi_n \Longleftrightarrow df^* = 0. \tag{3}$$

Furthermore, $\mathcal{Z}_n(\mathbb{Z}_2^n)$ is generated by small covers over products of simplices.

Our observation and Main theorem

- $X(\mathbb{Z}_2^n)$: Universal complex of $\mathbb{Z}_2^n$ whose simplices are subsets of $\mathbb{Z}_2^n$ which are linear indep.
- $\rho_1^* \cdots \rho_n^* \in \mathbb{Z}_2[\mathbb{Z}_2^n \setminus 0] \Leftrightarrow$ the $(n-1)$-simplex $[\rho_1^*, \ldots, \rho_n^*] \in X(\mathbb{Z}_2^n)$.
- the group $\mathcal{E}_n^* = \mathrm{Span}\{\tau^* \mid \tau$ faithful n-dim rep. $\} \Leftrightarrow$ the $(n-1)$-th chain group $C_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2)$
- the diff. $d \Leftrightarrow$ the boundary operator ∂ on $C_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2)$.
- $df^* = 0 \Leftrightarrow f^* \in Z_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2)$ is a closed $(n-1)$-cycle.
- since $X(\mathbb{Z}_2^n)$ is pure and $(n-1)$-dimensional, we have

Main theorem

$$\mathcal{Z}_n(\mathbb{Z}_2^n) \cong H_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2). \tag{4}$$

Universal complex $X(\mathbb{Z}_2^n)$ of $\mathbb{Z}_2^n$

UNIVERSAL SIMPLICIAL COMPLEXES INSPIRED BY TORIC TOPOLOGY

DJORDJE BARALIĆ, JELENA GRBIĆ, ALEŠ VAVPETIČ, ALEKSANDAR VUČIĆ

arxiv:1708.09565v2(2020)

- [Björner] Finite matroids are shellable complexes.
- [Björner] A pure n-dimensional shellable complex has the homotopy type of a wedge of n-spheres.
- $X(\mathbb{Z}_2^n)$ is a finite matroid, so $X(\mathbb{Z}_2^n) \simeq \bigvee_{A_n} S^{n-1}$.
- By Euler characteristic $\chi(X(\mathbb{Z}_2^n) = \sum_i (-1)^i f_i = 1 + (-1)^{n-1} A_n$,

$$A_n = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} \frac{(2^n - 2^i) \cdots (2^n - 1)}{(i+1)!} \tag{5}$$

dimension of $\mathcal{Z}_n(\mathbb{Z}_2^n)$

Corollary

$$\dim \mathcal{Z}_n(\mathbb{Z}_2^n) = A_n = (-1)^n + \sum_{i=0}^{n-1}(-1)^{n-1-i}\frac{(2^n-2^i)\cdots(2^n-1)}{(i+1)!}. \quad (6)$$

- $A_1 = 0$, $\mathcal{Z}_1(\mathbb{Z}_2) = 0$. ([Conner-Floyd] $\mathcal{Z}_*(\mathbb{Z}_2) = \mathbb{Z}_2$)
- $A_2 = 1$, $\mathcal{Z}_2(\mathbb{Z}_2^2) \cong \mathbb{Z}_2$. ([Conner-Floyd] $\mathcal{Z}_*(\mathbb{Z}_2^2) = \mathbb{Z}_2[[\mathbb{R}P^2]]$)
- $A_3 = 13$. ([Lü])
- $A_4 = 511$. ([Lü-Tan])
- $A_5 = 61,193$ (by Corollary above)
- ...

The whole story for bordism of 2-toric manifolds

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}_n(\mathbb{Z}_2^n) \arrow[r, "\phi_n"] \arrow[d, equal] & \mathcal{R}_n(\mathbb{Z}_2^n) & & & \\
0 \arrow[r] & \mathcal{Z}_n(\mathbb{Z}_2^n) \arrow[r, "\phi_n"] \arrow[d, equal] & \mathcal{E}_n \arrow[u, "\subseteq"'] & & & \\
0 \arrow[r] & \mathcal{Z}_n(\mathbb{Z}_2^n) \arrow[r, "\phi_n"] \arrow[d, equal] & \mathcal{E}_n^* \arrow[u, "\cong", "\text{daul basis}"'] \arrow[r, "d"] \arrow[d, equal] & \mathcal{E}_{n-1}^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{Z}_n(\mathbb{Z}_2^n) \arrow[r, "\phi_n"] & C_{n-1}(X(\mathbb{Z}_2^n)) \arrow[r, "\partial_{n-1}"] & C_{n-2}(X(\mathbb{Z}_2^n)) \arrow[r] & 0
\end{tikzcd}
$$

Outline

unitary T^n-manifolds

- *A unitary T^n-manifold* is an oriented closed smooth manifold equipped with an effective T^n-action, such that its tangent bundle admits a T^n-equivariant stable complex structure.
- *A unitary toric manifold* is a unitary T^n-manifold of dimension $2n$
- $\mathcal{Z}_m^U(T^n)$: the equivariant bordism group of m-dimensional unitary T^n-manifolds with finite fixed point set.
- $\mathcal{Z}_*^U(T^n) = \sum_m \mathcal{Z}_m^U(T^n)$: the equivariant bordism ring.

unitary T^n-manifolds

- M^{2n}: a unitary toric manifold
- $p \in M^T$ is a fixed point
- Two orientations on the tangent space T_pM:
  - the one induced from the orientation of M, and
  - the other induced from the stable complex structure on TM.
- Define the sign $\varepsilon(p)$ of p

$$\varepsilon(p) = \begin{cases} +1 & \text{if these two orientations coincide} \\ -1 & \text{otherwise} \end{cases} \tag{7}$$

- the tangent T^n-representaion at p can be decomposed to the product of irreducible T^n-representations

$$\tau_p = \tau_{p,1} \dots \tau_{p,n}$$

.

- [Masuda] $\{\tau_{p,1}, \dots, \tau_{p,n}\}$ are linearly independent

exterior algebra I

- $\mathrm{Hom}(T^n, S^1)(\cong \mathbb{Z}^n)$: set of all irreducible complex T^n-representations.
- $J_n^{\mathbb{C}} := \mathrm{Hom}(T^n, S^1) \setminus 0$.
- $\Lambda_{\mathbb{Z}}^*(J_n^{\mathbb{C}}) = \sum_m \Lambda_{\mathbb{Z}}^m(J_n^{\mathbb{C}})$: *exterior algebra* over $J_n^{\mathbb{C}}$
- *essential exterior monomial* $\tau = \tau_1 \wedge \cdots \wedge \tau_n$: $\{\tau_1, \cdots, \tau_m\} \subseteq J_n^{\mathbb{C}}$ is linearly independent.
- *essential exterior polynomial* $f = \sum \tau_{i,1} \wedge \cdots \wedge \tau_{i,m}$: sum of finitely many essential exterior T^n-monomial τ.
- $\mathcal{E}_m(J_n^{\mathbb{C}})$: the set of homogeneous essential exterior polynomials of degree $0 \leq m \leq n$.

monomorphism

- Rearrange the basis $\{\tau_{p,1}, \ldots, \tau_{p,n}\}$ if necessary such that

$$\det[\tau_{p,1} \ldots \tau_{p,n}] = \varepsilon(p)$$

- Define $\varphi_*^U : \mathcal{Z}_*^{\mathbb{C}}(T^n) \to \Lambda_{\mathbb{Z}}^*(J_n^{\mathbb{C}})$ to be

$$\varphi_*^U([M]) = \sum_{p \in M^{T^n}} \tau_{p,1} \wedge \cdots \wedge \tau_{p,n}$$

[Darby]

The map

$$\varphi_*^U : \mathcal{Z}_*^{\mathbb{C}}(T^n) \to \Lambda_{\mathbb{Z}}^*(J_n^{\mathbb{C}})$$

is a monomorphism as algebras over $\mathbb{Z}$.

exterior algebra II

Consider the dual of the exterior algebra $\Lambda_{\mathbb{Z}}(J_n^{\mathbb{C}})$

- $J_n^{*\mathbb{C}} := \mathrm{Hom}(\mathrm{Hom}(T^n, S^1), \mathbb{Z}) \setminus 0 \cong \mathbb{Z}^n \setminus 0$.
- $\Lambda_{\mathbb{Z}}^m(J_n^{*\mathbb{C}})$: *exterior polynomials* of degree m
- $\mathcal{E}_m(J_n^{*\mathbb{C}})$: *essential exterior T^n-polynomial* of degree m, where $0 \leq m \leq n$.
- $\tau_1 \wedge \cdots \wedge \tau_n \in \mathcal{E}_n(J_n^{\mathbb{C}}) \Leftrightarrow$
  $\tau_1^* \wedge \cdots \wedge \tau_n^* \in \mathcal{E}_n^*(J_n^{*\mathbb{C}}) \Leftrightarrow$
  $\{\tau_1, \ldots, \tau_n\}$ is a basis of $\mathrm{Hom}(\mathrm{Hom}(T^n, S^1), \mathbb{Z}) \cong \mathbb{Z}^n$

Darby's result

Define a differential on $\Lambda^*_{\mathbb{Z}}(J_n^{*\mathbb{C}})$

$$d_k(\tau_1\wedge\cdots\wedge\tau_k) = \begin{cases} \sum_{i=1}^{k}(-1)^{i+1}\tau_1\wedge\cdots\wedge\hat{\tau}_i\wedge\cdots\wedge\tau_k & \text{if } k>1 \\ 1 & \text{if } k=1,0 \end{cases} \tag{8}$$

- $d^2=0$, and $(\mathcal{E}_*(J_n^{*\mathbb{C}}), d)$ becomes a chain complex.

[Darby] Theorem 8.5

Let $g\in\mathcal{E}_n(J_n^{\mathbb{C}})$. Then $g\in \mathrm{Im}\varphi_n^U$ iff $d(g^*)=0$.

universal complex $X(\mathbb{Z}^n)$ of $\mathbb{Z}^n$

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- A subset $\{v_1, \ldots, v_m\}$ of $\mathbb{Z}^n$ is *unimodular* if $\mathrm{span}\{v_1, \ldots, v_m\}$ is a direct summand of $\mathbb{Z}^n$ with rank m.
- $X(\mathbb{Z}^n)$: the collection of all unimodular subsets of $\mathbb{Z}^n$. Specially, the vertices are corresponding to the primitive vectors in $\mathbb{Z}^n$.
- $X(\mathbb{Z}^n)$ is pure and $(n-1)$-dimensional.

X(Z^n) is homotopy equivalent to a countablely infinite wedge of $(n-1)$-spheres.

Our observation and Main theorem

- monomial $\rho_1^* \cdots \rho_n^* \in \mathcal{E}_n(J_n^{*\mathbb{C}}) \Leftrightarrow$ the $(n-1)$-simplex $[\rho_1^*, \ldots, \rho_n^*] \in X(\mathbb{Z}^n)$.
- $\mathcal{E}_n^*(J_n^{*\mathbb{C}}) = \mathrm{Span}\{\tau^* \mid \tau$ faithful n-dim complex rep. $\} \Leftrightarrow$ the $(n-1)$-th chain group $C_{n-1}(X(\mathbb{Z}^n), \mathbb{Z})$
- the diff. $d \Leftrightarrow$ the boundary operator ∂ on $C_{n-1}(X(\mathbb{Z}^n), \mathbb{Z})$.
- $df^* = 0 \Leftrightarrow f^* \in Z_{n-1}(X(\mathbb{Z}^n), \mathbb{Z})$ is a closed $(n-1)$-cycle.
- since $X(\mathbb{Z}^n)$ is pure and $(n-1)$-dimensional, we have

Main theorem

$$\mathcal{Z}_n(\mathbb{Z}^n) \cong Z_{n-1}(X(\mathbb{Z}^n), \mathbb{Z}) = H_{n-1}(X(\mathbb{Z}^n), \mathbb{Z}). \tag{9}$$

Corollary

$\mathcal{Z}_n^U(T^n) \cong H_{n-1}(X(\mathbb{Z}^n); \mathbb{Z})$ is a free abelian group with countable infinite rank.

The whole story for bordism of unitary toric manifolds

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}_n^U(T^n) \arrow[r, "\varphi_n^U"] \arrow[d, equal] & \Lambda_{\mathbb{Z}}^n(J_n^{\mathbb{C}}) & & \\
0 \arrow[r] & \mathcal{Z}_n^U(T^n) \arrow[r, "\varphi_n^U"] \arrow[d, equal] & \mathcal{E}_n(J_n^{\mathbb{C}}) \arrow[u, "\subseteq"'] & & \\
0 \arrow[r] & \mathcal{Z}_n^U(T^n) \arrow[r, "\varphi_n^U"] \arrow[d, equal] & \mathcal{E}_n(J_n^{*\mathbb{C}}) \arrow[u, "\cong"', "\text{daul basis}"] \arrow[r, "d"] \arrow[d, equal] & \mathcal{E}_{n-1}(J_n^{*\mathbb{C}}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{Z}_n^U(T^n) \arrow[r, "\varphi_n^U"] & C_{n-1}(X(\mathbb{Z}^n)) \arrow[r, "\partial_{n-1}"] & C_{n-2}(X(\mathbb{Z}^n)) \arrow[r] & 0
\end{tikzcd}
$$

Outline

simplicial spheres in universal complexes

- G_d is $\mathbb{Z}_2$ or S^1 and R_d is $\mathbb{Z}_2$ or $\mathbb{Z}$, as the notations in [DJ].
- S_d^{n-1}: simplicial $(n-1)$-sphere in universal complexe $X(R_d^n)$.
- Each maximal simplex of S_d^{n-1} consists of a basis of G_d^n.
- S_d^{n-1} is a simplicial sphere with a (injective) characteristic map $\lambda : V(S_d^{n-1}) \to R_d^n$.
- $M(S_d^{n-1}) = P_{S_d^{n-1}} \times G_d^n / \sim$. (construction in [DJ])
- $M(S_d^{n-1}) = \mathcal{Z}_{S_d^{n-1}} / \ker \tilde{\lambda}$, where $\tilde{\lambda} : R_d^{|V|} \to R_d^n$.
- $M(S_d^{n-1})$ is a dn-dimensional manifold.

Question

Is $M(S_d^{n-1})$ smooth?

If the answer is YES, then such manifold may be chosen to be representive element for equivariant bordism class.

small covers and equiviariant bordism classes

[Lü&Tan]:

- Theorem 2.6: $\mathcal{Z}_n(\mathbb{Z}_2^n)$ is generated by all general Bott tower which are small covers over product of simplices.
- Each class of $\mathcal{Z}_n(\mathbb{Z}_2^n)$ contains a small cover as its representative.

Problem

Search a new proof via the simplicial spheres in universal complex.

Observation:

- $\lambda : \partial P^* \to X(\mathbb{Z}_2^n)$ is a nondegerated simplical map, for any simple convex polytope P with char. map λ.
- Fact: every characteristic map on the product of simplices is an injection. $\lambda : (\partial \Pi_i \Delta^i)^* \to X(\mathbb{Z}_2^n)$ is an embedding.
- $\forall$ maximal simplexes σ and τ and $\sigma \cap \tau = \emptyset$, $\exists\ \gamma$, such that $\gamma \subseteq \sigma \cup \tau$ as subsets of $\mathbb{Z}_2^n$.
- Hence, the maxiaml simplexes of 'essential' sphere maybe intersect to each other.

omniorientation

- S^{n-1}: simplicial $(n-1)$-sphere in $X(\mathbb{Z}^n)$.
- omniorientation of S^{n-1}: consists of two choices
  - an orientation for S^{n-1}
  - and orientations for every link $link_S(v)$ for all vertices $v \in V(S^{n-1})$
- omniorientation of $S^{n-1} \cong$ omniorientation of $P_{S^{n-1}}$
- omniorientation of S^{n-1} determines an omniorientation of $M(S^{n-1})$
- If $M(S^{n-1})$ is smooth, then $M(S^{n-1})$ is a unitary torous manifold.
- Thanks to the Steinitz theorem, every $M(S^2)$ is a 6-dim quasitoric manifold.

$\mathcal{Z}_3(T^3)$ is generated by quasitoric manifolds.

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