Cohomological rigidity for Bott manifolds

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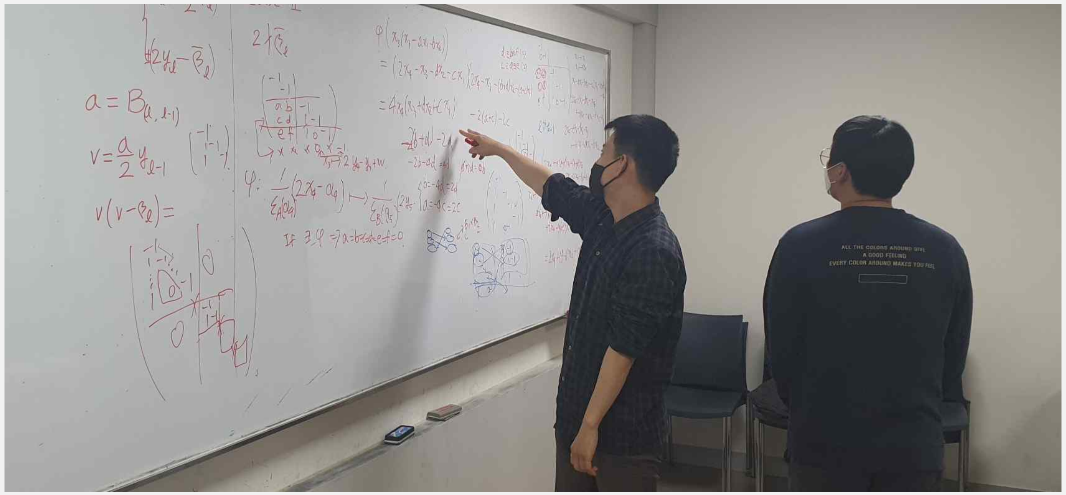
Reference

This work is based on the paper

"Strong Cohomological Rigidity of Bott manifolds" (arXiv:2202.10920)

jointly with Taekgyu Hwang and Hyeontae Jang.



Cohomology as invaraint

$f : X \to Y$ differentiable	$\Rightarrow$	$f : X \to Y$ continuous	$\Rightarrow$	$f^* : H^*(Y) \to H^*(X)$ homomorphism
diffeomorphism		homeomorphism		isomorphism

Reverse direction

$$X \xrightarrow{\text{df eo}} Y \quad \overset{?}{\Longleftarrow} \quad X \xrightarrow{\text{hom eo}} Y \quad \overset{?}{\Longleftarrow} \quad H^*(Y) \xrightarrow{\text{iso}} H^*(X)$$

- Poincaré conjecture
- Borel conjecture
- …

Hirzebruch surfaces

For $a \in \mathbb{Z}$, the Hirzebruch surface Σ_a is defined as

$$\Sigma_a = \mathbb{P}(\mathbb{C} \oplus \gamma^{ax_1}) \to \mathbb{C}P^1,$$

where γ^{ax_1} is the complex line bundle over $\mathbb{C}P^1$ satisfying

$$c_1(\gamma^{ax_1}) = ax_1 \in H^*(\mathbb{C}P^1) = \frac{\mathbb{Z}[x_1]}{x_1^2}.$$

Bott tower

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}$$

- $B_i = \mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_i}) \rightarrow B_{i-1}$
- γ^{α_i} is the complex line bundle over B_{i-1} s.t. $c_1(\gamma^{\alpha_i}) = \alpha_i$

Example

- B_2 : Hirzebruch surface
- If all π_i are trivial, $B_n \cong \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$.

B_n : an n-stage Bott manifold

Classification of Hirzebruch surfaces

Note that $H^*(\Sigma_a) = \frac{\mathbb{Z}[x_1,x_2]}{x_1^2\,,\ x_2^2 - a x_1 x_2}$ $(\deg x_i = 2)$. One can see that

$$H^*(\Sigma_a) \cong H^*(\Sigma_b) \iff a \equiv b \pmod 2.$$

Set $b = a + 2k$.

$$\Sigma_a = \mathbb{P}(\mathbb{C} \oplus \gamma^{a x_1}) \cong \mathbb{P}\left(\gamma^{k x_1} \otimes (\mathbb{C} \oplus \gamma^{a x_1})\right)$$

$$\cong \mathbb{P}\left(\gamma^{k x_1} \oplus \gamma^{(a+k) x_1}\right) \cong \mathbb{P}\left(\mathbb{C} \oplus \gamma^{(a+2k) x_1}\right) = \Sigma_b$$

$$\boxed{\left(\because\ c\left(\gamma^{k x_1} \oplus \gamma^{(a+k) x_1}\right) = c\left(\mathbb{C} \oplus \gamma^{(a+2k) x_1}\right) = (a+2k) x_1 \in H^*(\mathbb{C}P^1)\right)}$$

Classification

Theorem (Hirzebruch 1951)

$$H^*(\Sigma_a) \cong H^*(\Sigma_b) \iff \Sigma_a \cong \Sigma_b$$

Theorem (Masuda-Panov 2008)

$$H^*(B_n) \cong H^*((\mathbb{C}P^1)^n) \Rightarrow B_n \cong (\mathbb{C}P^1)^n$$

Cohomological rigidity problem

Cohomological Rigidity Conjecture for Bott manifolds

$$H^*(B_n) \cong H^*(B_n') \overset{?}{\Longrightarrow} B_n \cong B_n'$$

Strong CR Conjecture for Bott manifolds

$$\varphi : H^*(B_n) \xrightarrow{\text{iso}} H^*(B_n') \quad \text{as graded rings}$$

$$\overset{?}{\Longrightarrow} \quad \exists f : B_n' \xrightarrow{\text{df}} B_n \quad \text{s.t.} \quad \varphi = f^*$$

Toric manifolds

- A toric manifold is a complete non-singular toric variety.
- An n-stage Bott manifold B_n is known to be a toric manifold over an n-cube I^n.

CR Problem for Toric manifolds

$$H^*(X) \cong H^*(Y) \quad \Rightarrow \quad X \cong Y$$

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Strong CR Problem for Toric manifolds

$$\varphi : H^*(X) \xrightarrow{\text{iso}} H^*(Y) \quad \text{as graded rings}$$

$$\Rightarrow \quad \exists f : Y \xrightarrow{\text{df}} X \quad \text{s.t.} \quad \varphi = f^*$$

Negative

Cohomology of Bott manifolds

- $H^*(B_i) = \frac{H^*(B_{i-1})[x_i]}{x_i^2 - \alpha_i x_i}$ $(\deg x_i = 2)$ $\boxed{B_i = \mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_i}) \to B_{i-1}}$

- $H^*(B_n) = \frac{\mathbb{Z}[x_1, \ldots, x_n]}{x_i^2 - \alpha_i x_i,\ i=1,\ldots,n}$, where $\alpha_i = a_{i,1}x_1 + \cdots + a_{i,i-1}x_{i-1}$

$$B_n(A) := B_n \quad \longleftrightarrow \quad A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{2,1} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Bott matrix

Stability

$$F_k(A) \coloneqq \mathrm{span}\{\, x_1 \,, \dots , x_k\} \subset \; H^2(B_n(A))$$

A graded ring isomorphism $\varphi : H^*\big(B_n(A)\big) \to H^*\big(B_n(B)\big)$ is called k-stable if $\varphi\big(F_k(A)\big) \subset F_k(B)$.

Partial Results (for Bott manifolds)

- Choi-Masuda-Suh (2010) CR holds for 3-stage Bott
- Choi-Masuda (2012) SCR holds for $\mathbb{Q}$-trivial Bott
- Ishida (2012) SCR holds if φ is (n-1)-stable
- Choi-Masuda-Murai (2015) Pontragin class is preserved by φ
- Choi (2015) CR holds for 4-stage Bott
- Higashitani-Kurimoto (2022+) CR holds for Fano Bott
- Ishida (2022+) SCR holds if φ is (n-2)-stable

Theorem (Ishida 2012)

Any $(n-2)$-stable cohomology ring iso is realizable by a diffeo.

Main Results

Theorem (Choi-Hwang-Jang 2022+)

Strong Cohomological Rigidity holds for Bott manifolds.

$\varphi : H^*\big(B_n(A)\big) \;\to\; H^*(B_n(B))$ graded ring iso.

Then, $\exists$ A' and B' s.t. $B_n(A) \underset{f}{\to} B_n(A'), B_n(B) \underset{g}{\to} B_n(B')$ and

$$\psi = (g^{-1})^* \circ \varphi \circ f^* : H^*\big(B_n(A')\big) \;\to\; H^*(B_n(B'))$$

is either $(n-1)$ or $(n-2)$ -stable.

We will use an induction.

$\varphi : H^*(B_n(A)) \;\to\; H^*(B_n(B))$ graded ring iso.

Then, $\exists\, A'$ and B' s.t. $B_n(A) \underset{f}{\to} B_n(A'), B_n(B) \underset{g}{\to} B_n(B')$ and

$$\psi = (g^{-1})^* \circ \varphi \circ f^* : H^*(B_n(A')) \;\to\; H^*(B_n(B'))$$

is either $(n-1)$ or $(n-2)$ -stable.

① $\exists\; A', B'$ s.t. ψ is $k\ (>0)$-stable.

② If $\varphi : H^*(B_n(A)) \;\to\; H^*(B_n(B))$ is k-stable, then

$\exists\; A', B'$ s.t. ψ is either $(k+1)$ or $(k+2)$-stable.

Two operations

Rank 2 decomposable vector bundles over a Bott manifold are classified by their Chern classes. (Ishida 2012)

(Twisting) Suppose $v \in F_{j-1}(A)$ satisfying $v(\alpha_j - v) = 0$

$$\mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_j}) \cong \mathbb{P}\left(\gamma^{-v} \otimes (\mathbb{C} \oplus \gamma^{\alpha_j})\right)$$

$$\cong \mathbb{P}(\gamma^{-v} \oplus \gamma^{\alpha_j - v}) \cong \mathbb{P}\left(\mathbb{C} \oplus \gamma^{\alpha_j - 2v}\right)$$

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{j-1} \\ \alpha_j \\ \alpha_{j+1} \\ \vdots \\ \alpha_n \end{pmatrix} \longrightarrow A' = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{j-1} \\ \alpha_j - 2v \\ \alpha_{j+1} - a_{j+1,j}v \\ \vdots \\ \alpha_n - a_{n,j}v \end{pmatrix}$$

Two operations

(Switching) Suppose $a_{j+1,j} = 0$.

$$
\begin{tikzcd}
\mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_{j+1}}) \arrow[r] \arrow[d, equal] & \mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_j}) \arrow[r] & B_{j-1} \\
\mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_j}) \arrow[r] & \mathbb{P}(\mathbb{C} \oplus \gamma^{\alpha_{j+1}}) \arrow[r] & B_{j-1}
\end{tikzcd}
$$

A' is obtained from A by

$$
\begin{aligned}
j\text{th row} &\longleftrightarrow (j+1)\text{th row} \\
j\text{th column} &\longleftrightarrow (j+1)\text{th column}
\end{aligned}
$$

$\mathbb{Q}$-trivial Bott manifold

- B_n is $\mathbb{Q}$-trivial if

$$H^*(B_n;\mathbb{Q}) \cong H^*(\,(\mathbb{C}P^1)^n;\mathbb{Q}) \cong \frac{\mathbb{Q}[x_1,\dots,x_n]}{x_1^2,\dots,x_n^2}$$

Theorem (Choi-Masuda 2012)

TFAE :

(1) B_n is $\mathbb{Q}$-trivial

(2) $\alpha_i^2 = 0$ for $i = 1,\dots,n$

(3) only n square zero elts in $H^2(B_n)$ up to scalar multiplication;

$$2x_i - \alpha_i \quad \text{for } i = 1,\dots,n$$

Observation

By Twistings and Switchings

We can make $B_n(A)$ well-ordered; it implies that $B_n(A)$ has a fibration structure

$$B_{n-k}(\bar{A}) \to B_n(A) \to B_k(\hat{A}),$$

where $B_{n-k}(\bar{A})$ is $\mathbb{Q}$-trivial.

$$A = \begin{pmatrix} \hat{A} & 0 \\ * & \bar{A} \end{pmatrix}$$

The minimal number k_0 (> 0) is a ring invariant.

Proof of the main theorem

- Assume that $B_n(A)$ and $B_n(B)$ are well-ordered, and
$$\varphi : H^*\big(B_n(A)\big) \to H^*(B_n(B))$$

- φ is $k_0(>0)$-stable. ① Done
- $\bar{\varphi} = \varphi|_{\frac{H^*(B_n)}{H^*(B_k)}}: H^*\big(B_{n-k}(\bar{A}\,)\big) \to H^*\big(B_{n-k}(\bar{B}\,)\big)$ is an isomorphism.

From now on, we will take Twisting and Switching only for $j \geq k_0$. Then, it does not break the well-ordered structure.
Hence, the fiber $B_{n-k}(\bar{A}\,)$ is $\mathbb{Q}$-trivial.
We assume that φ is k-stable.

- $H^*\left(B_{n-k}(\bar{A}\,)\right) = \frac{H^*(B_n)}{H^*(B_k)} = \frac{\mathbb{Z}[\bar{x}_{k+1},\ldots,\bar{x}_n]}{\bar{x}_i^2=\bar{\alpha}_i\,\bar{x}_i\,, i=k+1,\ldots,n}$

- $\bar{\varphi}(2\bar{x}_t - \bar{\alpha}_t) = a(\,2\bar{y}_{\sigma(t)} - \bar{\beta}_{\sigma(t)})$, for some a and $\sigma \in \mathcal{S}_{\{k+1,\ldots,n\}}$.

Suppose $\sigma(k+1) = \ell$.

Equivalently, $\varphi(x_{k+1}) = a(\,2y_{\ell} - \beta_{\ell}) + \omega$ for some $\omega \in F_k(B)$

- If $\sigma(k+1) = k+1$, then φ is $(k+1)$-stable, as desired.

- If $\sigma(k+1) = \ell > k+1$, set $\beta_\ell = p\, y_{\ell-1} + \gamma$
- Case 1 : if $p \equiv 0 \pmod{2}$, $$\boxed{\frac{p}{2} y_{\ell-1}\left(\beta_\ell - \frac{p}{2} y_{\ell-1}\right) = 0}$$

Hence, we apply both Twisting and Switching to make

$$\sigma(k+1) = \ell - 1.$$

- If $\sigma(k+1) = \ell > k+1$, set $\beta_\ell = p\, y_{\ell-1} + \gamma$
- Case 2 : if $p \equiv 1 \pmod{2}$, $\boxed{\dfrac{\bar{\beta}_{\ell-1}}{2}\left(\beta_\ell - \dfrac{\bar{\beta}_{\ell-1}}{2}\right) = 0}$

Hence, we apply both Twisting and Switching to make

$$\sigma(k+1) = \ell - 1 \text{ or } \ell - 2$$

Complete of the proof

Consider $\varphi : H^*\big(B_n(A)\big) \;\to\; H^*(B_n(B))$ and φ^{-1}.

Set the corresponding permutations σ and η in $\mathcal{S}_{\{k+1,\ldots,n\}}$.

We may assume $\sigma(k+1), \eta(k+1) \leq k+3$ by cases 1 and 2.

Since switching changes both σ and η simultaneously, we have to avoid some cases in the case $p \equiv 1 \pmod{2}$.

$\mathbb{Q}$-trivial Bott manifold

- $\mathcal{H}_n = B_n(A_n)$ with

$$A_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

Theorem (Choi-Masuda 2012)

$$B_n \text{ is } \mathbb{Q}\text{-trivial} \iff B_n \cong \mathcal{H}_{n_1} \times \cdots \times \mathcal{H}_{n_\ell}$$

By Twistings and Switchings

$\mathbb{Q}$-trivial Bott manifold

- $B_n(A)$ is $\mathbb{Q}$-trivial

$$A \to \begin{pmatrix} A_{n_1} & 0 & \cdots & 0 \\ 0 & A_{n_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_\ell} \end{pmatrix}$$

By Twistings and Switchings

Each row of A corresponds to a component of $(n_1, n_2, \ldots, n_\ell) \vdash n$.
Two rows are in the same block if they correspond the same component.

$\mathbb{Q}$-trivial Bott manifold

- $B_n(A)$ is $\mathbb{Q}$-trivial

$$A \to \begin{pmatrix} A_{n_1} & 0 & \cdots & 0 \\ 0 & A_{n_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_\ell} \end{pmatrix}$$

By Twistings and Switchings

Each row of A corresponds to a component of $(n_1, n_2, \dots, n_\ell) \vdash n$.
Two rows are in the same block if they correspond the same component.

Observation

- If $(k+1)$ and $(k+2)\text{th}$ rows are in the same block in A, then so are $\sigma(k+1)$ and $\sigma(k+2)$ in B.
- If $p \equiv 1 \pmod 2$, then ℓ and $\ell+1$ are in the same block in B.

These make some restrictions of σ and η that enable us to avoid the unexpected cases.

Hence, one can apply Switching and Twisting again to obtain

$$\sigma(k+1) = \eta(k+1) = k+1 \text{ or } k+2$$

② Done

Main Results

Theorem (Choi-Hwang-Jang 2022+)

Strong Cohomological Rigidity holds for Bott manifolds.

Hasui-Kuwata-Masuda-Park (2020) studied the CR of toric manifolds over $vc\,(I^n)$, assuming the CR for Bott manifolds.

Corollary

Toric manifolds over $vc\,(I^n)$ are cohomologically rigid.