

Seshadri constants of equivariant vector bundles on toric varieties

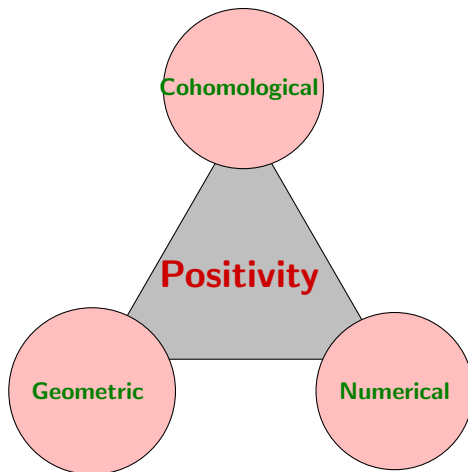
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- “Positivity” of line bundles means that it has “many global sections”.



Positivity of line bundles

Framework: All varieties are nonsingular projective defined over \mathbb{C} .

Let \mathcal{L} be a line bundle on a variety X and s_0, s_1, \dots, s_N be a \mathbb{C} -basis for $H^0(X, \mathcal{L})$. Then there is the associated **Kodaira map**

$$\phi_{\mathcal{L}} : X \setminus Bs(\mathcal{L}) \longrightarrow \mathbb{P}^N, \text{ defined by } x \longmapsto [s_0(x) : s_1(x) : \dots : s_N(x)],$$

where $Bs(\mathcal{L}) := \mathbb{V}(s_0) \cap \dots \cap \mathbb{V}(s_N)$ is the base locus of the line bundle \mathcal{L} .

- The line bundle \mathcal{L} is called **globally generated** if $Bs(\mathcal{L}) = \emptyset$. In addition, if $\phi_{\mathcal{L}}$ defines a closed embedding $\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}^N$, then \mathcal{L} is said to be **very ample**.
- The line bundle \mathcal{L} is called **ample** if there exists a positive integer m such that $\mathcal{L}^{\otimes m}$ is very ample.

Some criteria for ampleness

Theorem 1 (Nakai-Moishezon-Kleiman criterion)

Let \mathcal{L} be a line bundle on a projective variety X . Then \mathcal{L} is ample if and only if

$$\mathcal{L}^{\dim V} \cdot V > 0$$

for every positive dimensional irreducible subvariety $V \subseteq X$.

- A line bundle \mathcal{L} is called **numerically effective (nef)** if $\mathcal{L} \cdot C \geq 0$ for all irreducible curves C in X .

Seshadri criterion for ampleness (1972)

A line bundle \mathcal{L} on X is ample if and only if for every point $x \in X$ there exists a positive number ε such that $\frac{\mathcal{L} \cdot C}{\text{mult}_x C} \geq \varepsilon$ for all curves C passing through x .

Lets look for optimal values of ε !

Definition 1 (Demailly(1992))

Let \mathcal{L} be a nef line bundle on a complex projective variety X . For a point $x \in X$, the Seshadri constant of \mathcal{L} at x is defined to be

$$\varepsilon(X, \mathcal{L}, x) := \inf_{x \in C} \frac{\mathcal{L} \cdot C}{\text{mult}_x C}.$$

This numerical invariant measures the "local positivity" of the line bundle \mathcal{L} at the point x .

Reformulation of Seshadri's ampleness criterion

A nef line bundle on X is ample if and only if $\varepsilon(\mathcal{L}) := \inf_{x \in X} \varepsilon(X, \mathcal{L}, x) > 0$.

Original goal

Fujita conjecture

Let X be a projective variety. Let \mathcal{L} be an ample line bundle on X and $n = \dim(X)$.

- $K_X + m\mathcal{L}$ is globally generated for $m \geq n + 1$.
- $K_X + m\mathcal{L}$ is very ample for $m \geq n + 2$.

The result is known for $n = 2$ ([Rider \(1988\)](#)) and $n = 3$ ([Ein-Lazarsfeld](#)).

Demailly's approach

- If $\varepsilon(X, \mathcal{L}, x) > \frac{n}{n+1}$ for all $x \in X$ then $K_X + (n+1)\mathcal{L}$ is globally generated.
- If $\varepsilon(X, \mathcal{L}, x) > \frac{2n}{n+2}$ for all $x \in X$ then $K_X + (n+2)\mathcal{L}$ is very ample.

[Miranda\(1993\)](#) : too optimistic to conclude Fujita conjecture.

Miranda's Example

Fix any $\delta > 0$, then there exists a smooth surface X , a point $x \in X$, and an ample line bundle \mathcal{L} on X such that

$$\varepsilon(X, \mathcal{L}, x) < \delta.$$

- Recall $\varepsilon(\mathcal{L}) := \inf_{x \in X} \varepsilon(X, \mathcal{L}, x)$.
- From Miranda's example, we know that $\varepsilon(\mathcal{L})$ can be arbitrarily small.
- If \mathcal{L} is very ample, then $\varepsilon(\mathcal{L}) \geq 1$.
- Define $\varepsilon(X) := \inf_{\mathcal{L} \text{ ample}} \varepsilon(\mathcal{L})$.
- Is $\varepsilon(X) = 0$ for some X ?

Recently, Shripad M. Garge and Arghya Pramanik (arXiv:2202.08074) have answered this question by constructing some examples.

Some more bounds

- Define $\varepsilon(\mathcal{L}, 1) := \sup_{x \in X} \varepsilon(X, \mathcal{L}, x)$.
- **Ein-Lazarsfeld (1993)**: Let X be a smooth projective surface, and \mathcal{L} be an **ample** line bundle on X . Then

$$\varepsilon(X, \mathcal{L}, x) \geq 1$$

for “very general point” $x \in X$. Hence $\varepsilon(\mathcal{L}, 1) \geq 1$.

- **Lazarsfeld’s conjecture**: Let X be a nonsingular projective variety and \mathcal{L} be an ample line bundle on X , then $\varepsilon(\mathcal{L}, 1) \geq 1$.
- **Oguiso (2002)**:
 - $\varepsilon(\mathcal{L}, 1) = \varepsilon(X, \mathcal{L}, x)$ for very general $x \in X$.
 - $\varepsilon(\mathcal{L}) = \varepsilon(X, \mathcal{L}, x)$ for “special” $x \in X$.
- For any point $x \in X$, we have

$$0 \leq \varepsilon(\mathcal{L}) \leq \varepsilon(X, \mathcal{L}, x) \leq \varepsilon(\mathcal{L}, 1) \leq \sqrt[n]{\mathcal{L}^n}.$$

What to look for

Guiding problems on Seshadri constants

- Computing Seshadri constants.
- Giving bounds on them.
- Checking if they are irrational (Nagata Conjecture -1958).

Let us look at some existing results and applications in this context:

- Let \mathcal{L} be an **ample and globally generated** line bundle on a variety X , then $\varepsilon(X, \mathcal{L}, x) \geq 1$ for all $x \in X$.
- **Characterization of \mathbb{P}^n :**
Bauer-Szemberg (2009): Let X be a smooth Fano variety of dimension n . Then $X = \mathbb{P}^n \iff \varepsilon(X, -K_X, x) \geq n + 1$ for some $x \in X$.
- **DiRocco (1999)** has computed Seshadri constants of an ample line bundle over a toric variety at any fixed point.
- **Ito (2014)** has given bounds on Seshadri constants on an **arbitrary toric variety** at any point.

Seshadri constant for vector bundles

X : nonsingular complex projective variety, \mathcal{E} : vector bundle on X

$\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$: projectivized bundle associated to \mathcal{E}

$\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: tautological line bundle on $\mathbb{P}(\mathcal{E})$

A vector bundle \mathcal{E} on X is ample (resp. nef) if the tautological line bundle ξ is ample (resp. nef) on the projectivized bundle $\mathbb{P}(\mathcal{E})$.

Definition 2 (Hacon (2000), Fulger-Murayama (2021))

The Seshadri constant of a nef vector bundle \mathcal{E} at $x \in X$ is defined to be

$$\varepsilon(X, \mathcal{E}, x) := \inf_{C \subset \mathbb{P}(\mathcal{E})} \frac{\xi \cdot C}{\text{mult}_x \pi_* C},$$

where the infimum is taken over all curves C on $\mathbb{P}(\mathcal{E})$ that meet $\pi^{-1}(x)$ but not completely contained in $\pi^{-1}(x)$.

Some known results

- Let \mathcal{E} be an **ample and globally generated** vector bundle on a smooth complex projective curve X , then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) \geq 1.$$

- **Another Characterization of \mathbb{P}^n :** Let X be a smooth Fano variety of dimension n with nef tangent bundle. Then

$$X = \mathbb{P}^n \iff \varepsilon(X, \mathcal{T}_X, x) > 0 \text{ for some } x \in X,$$

(Fulger-Murayama (2021)).

Some special cases

- **Hacon (2000)**: Let \mathcal{E} be a nef vector bundle on a smooth complex projective curve X , then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \mu_{\min}(\mathcal{E}),$$

where $\mu_{\min}(\mathcal{E})$ denotes the smallest slope of any quotient bundle of \mathcal{E} .

Here slope of the vector bundle \mathcal{E} is $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})}$.

- **Fulger-Murayama (2021)**: If $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ is a nef vector bundle on a variety X , then for any $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \min_{1 \leq i \leq r} \{\varepsilon(X, \mathcal{E}_i, x)\}.$$

- **Fulger-Murayama (2021)**: \mathcal{E} **semistable discriminant zero** nef vector bundle of rank r on a variety X , then for all $x \in X$,

$$\varepsilon(X, \mathcal{E}, x) = \frac{1}{r} \varepsilon(X, \det(\mathcal{E}), x).$$

Definition 3

A *toric variety* X : A normal complex variety which contains a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset such that:

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ \downarrow & & \downarrow \\ T \times X & \longrightarrow & X \end{array}$$

Example 4

- $(\mathbb{C}^*)^n$, \mathbb{C}^n and \mathbb{P}^n .

Theorem 2 (Fundamental theorem for toric varieties)

The category of *toric varieties* is *equivalent* to the category of *fans*.

$$X_{\Delta} \longleftrightarrow \Delta_X.$$

Combinatorics of toric varieties

Combinatorial Data: $M = \text{Hom}(T, \mathbb{C}^*)$,
 $N = \text{Hom}(M, \mathbb{Z})$, fan Δ in $N \otimes \mathbb{R} \cong \mathbb{R}^n$.

- Cone $\sigma \in \Delta \rightsquigarrow$ affine variety U_σ ,
distinguished point $x_\sigma \in U_\sigma$.
- x_σ is a torus fixed point $\Leftrightarrow \sigma \in \Delta$ is
 n -dimensional.
- 1-dimensional cone $\rho \in \Delta$
 \rightsquigarrow invariant divisors D_ρ .
- $(n-1)$ -dimensional cone $\tau \in \Delta$
 \rightsquigarrow invariant curves $V(\tau) \cong \mathbb{P}^1$.

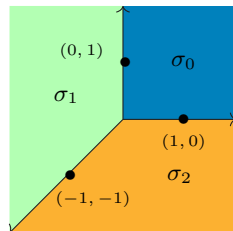


Figure: Fan of \mathbb{P}^2

Toric vector bundle

A T -equivariant vector bundle or toric vector bundle: A vector bundle $\pi : \mathcal{E} \rightarrow X$ on X with a lift of the action of T on the total space \mathcal{E} in such a way that:

- 1 the projection map π is equivariant, i.e., for all $e \in \mathcal{E}$ and $t \in T$ the following diagram commutes:

$$\begin{array}{ccc} T \times \mathcal{E} & \xrightarrow{\quad} & \mathcal{E} \\ \downarrow Id \times \pi & \begin{array}{c} (t, e) \mapsto t \cdot e \\ \downarrow \end{array} & \downarrow \pi \\ T \times X & \xrightarrow{\quad} & X \\ & \begin{array}{c} (t, \pi(e)) \mapsto t \cdot \pi(e) \\ \downarrow \end{array} & \end{array}$$

- 2 the torus T acts linearly on the fibers.

Example 5

line bundle, tangent bundle, cotangent bundle

Klyachko's classification theorem

\mathcal{E} : rank r toric vector bundle on X , $E = \mathcal{E}(1_T)$: the fiber at $1_T \in T \subset X$.

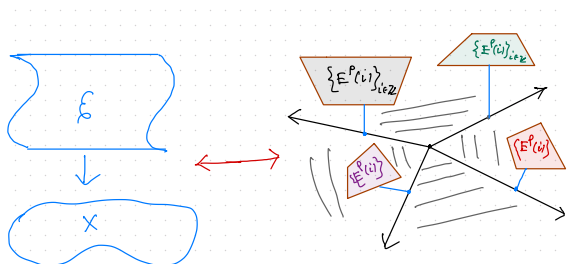
Klyachko (1990)

$$\mathcal{E} \longleftrightarrow E \supset \dots \supset E^\rho(i) \supset E^\rho(i+1) \supset \dots \mathbf{0},$$

rays of Δ satisfying compatibility condition:

for any $\sigma \in \Delta$, there exists a \hat{T}_σ -grading $E = \bigoplus_{\chi \in \hat{T}_\sigma} E^{[\sigma]}(\chi)$, such that

$$E^\rho(i) = \bigoplus_{\langle \chi, v_\rho \rangle \geq i} E^{[\sigma]}(\chi) \text{ for all } \rho \in \sigma(1).$$



- X toric variety; $x \in X$ a **torus fixed point** and \mathcal{E} a nef toric vector bundle on X . Then

$$\varepsilon(X, \mathcal{E}, x) = \min \{ \mu_{\min}(\mathcal{E}|_C) \mid x \in C \text{ and } C \text{ is an invariant curve} \}$$

([Hering-Mustață-Payne \(2010\)](#)).

Goal: To compute Seshadri constant at arbitrary points.

Recall: to compute Seshadri constant at $x \in X$, we need to compute the ratios

$$\frac{\xi \cdot C}{\text{mult}_x \pi_* C}, \quad \text{for all } C \subset \mathbb{P}(\mathcal{E}).$$

Key ingredient: the description of the Mori cone $\overline{\text{NE}}(\mathbb{P}(\mathcal{E}))$: the closed cone of curves of the projective bundle $\mathbb{P}(\mathcal{E})$.

X : toric variety; \mathcal{E} : toric vector bundle on X ; l_1, \dots, l_m : invariant curves in X .

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}|_{l_i}) & \xhookrightarrow{\eta_i} & \mathbb{P}(\mathcal{E}) \\ \pi_i \downarrow & & \downarrow \pi \\ l_i & \xhookrightarrow{\quad} & X \end{array}$$

- Since $\mathbb{P}(\mathcal{E}|_{l_i})$ is a toric variety, there is an invariant fiber curve Σ_i and invariant section curve Ω_i such that $\overline{\text{NE}}(\mathbb{P}(\mathcal{E}|_{l_i})) = \text{Cone}(\Sigma_i, \Omega_i)$.

Proposition 6 (Hering-Mustařa-Payne (2010))

Take $C_0 := \eta_i(\Sigma_i)$ and $C_i := \eta_i(\Omega_i)$, then the Mori cone is given by

$$\overline{\text{NE}}(\mathbb{P}(\mathcal{E})) = \left\{ a_0 C_0 + \dots + a_m C_m \mid a_i \in \mathbb{R}_{\geq 0} \text{ for } i = 0, \dots, m \right\}.$$

In particular, $\overline{\text{NE}}(\mathbb{P}(\mathcal{E}))$ is a polyhedral cone.

Theorem 7 (— - Khan - Aditya)

Let \mathcal{E} be a “**nice**” nef equivariant vector bundle of rank r on the projective space $X = \mathbb{P}^n$ ($n \geq 2$). Then for any point $x \in X$, we have

$$\varepsilon(\mathcal{E}, x) = \min_{1 \leq i \leq m} \{ \mu_{\min}(\mathcal{E}|_{l_i}) \}.$$

Example 8

- Uniform bundle: a bundle of splitting type (a_1, \dots, a_r) , i.e., for any line $l \subset \mathbb{P}^n$, we have

$$\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r).$$

- $\mathcal{T}_{\mathbb{P}^n}$ is a uniform bundle with splitting type $(2, 1, \dots, 1)$, hence for any $x \in \mathbb{P}^n$ the Seshadri constant is given by

$$\varepsilon(\mathcal{T}_{\mathbb{P}^n}, x) = 1.$$

Hirzebruch surface

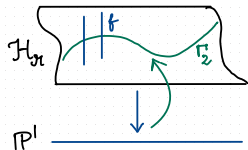


Figure: $\mathcal{H}_{c_{1,2}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c_{1,2}))$

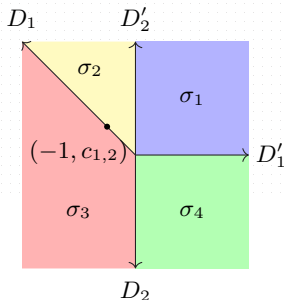


Figure: Fan for $\mathcal{H}_{c_{1,2}}$

- We have $D_1 \equiv D'_1 \equiv f$,
 $D'_2 \equiv D_2 - c_{1,2} D_1 \equiv \Gamma_2$
- The Picard group is
 $\text{Pic}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$.
- The Nef cone is
 $\text{Nef}(X) = \mathbb{R}_{\geq 0}D_1 \oplus \mathbb{R}_{\geq 0}D_2$,
assuming $c_{1,2} \geq 0$.
- The Mori cone is
 $\overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}\Gamma_2 \oplus \mathbb{R}_{\geq 0}f$.

Theorem 9 (___ - Khan- Aditya)

Let \mathcal{E} be an equivariant nef vector bundle of rank r on the Hirzebruch surface $X_2 = \mathcal{H}_{c_1,2}$ satisfying the following conditions:

$$\mu_{\min}(\mathcal{E}|_{D_1}) = \mu_{\min}(\mathcal{E}|_{D'_1}) \text{ and } \mu_{\min}(\mathcal{E}|_{D_2}) \geq \mu_{\min}(\mathcal{E}|_{D_1}).$$

Then for any $x \in X_2$, the Seshadri constant is given by:

$$\varepsilon(X_2, \mathcal{E}, x) = \begin{cases} \min\{\mu_{\min}(\mathcal{E}|_{D_1}), \mu_{\min}(\mathcal{E}|_{D'_2})\}, & \text{if } x \in \Gamma_2, \\ \mu_{\min}(\mathcal{E}|_{D_1}), & \text{if } x \notin \Gamma_2. \end{cases}$$

Seshadri constants of line bundles on Hirzebruch surfaces have been computed by [Syzdek \(2005\)](#), [García \(2006\)](#), [Hanumanthu-Mukhopadhyay \(2017\)](#).

Example 10

Consider the tangent bundle $\mathcal{E} = \mathcal{T}_{X_2}$ on the Hirzebruch surface X_2 . Then the associated filtrations $(E, \{E^i(j)\}_{i=1, \dots, 4; j \in \mathbb{Z}})$ are given by:

$$E^i(j) = \begin{cases} \mathbb{C}^2 & j \leq 0 \\ \text{Span}(v_i) & j = 1 \\ 0 & j > 1 \end{cases}.$$

$\mathcal{E} \otimes \mathcal{O}(D)$ is nef, where $D = a_1 D_1 + a_2 D_2$, $a_1 \geq c_{1,2}$, $a_2 \geq 0$.

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D'_1} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2), \quad (\mathcal{E} \otimes \mathcal{O}(D))|_{D_1} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2),$$

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D'_2} = \mathcal{O}_{\mathbb{P}^1}(a_1 - c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_1),$$

$$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_2} = \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2} a_2 + c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2} a_2 + 2).$$

The Seshadri constant is given by

$$\varepsilon(\mathcal{E} \otimes \mathcal{O}(D), x) = \begin{cases} \min\{a_1 - c_{1,2}, a_2\}, & \text{if } x \in \Gamma_2, \\ a_2, & \text{if } x \notin \Gamma_2. \end{cases}$$

Bott towers

Bott towers are a particular class of nonsingular projective toric varieties. They were constructed by [Grossberg-Karshon \(1994\)](#).

For an integer $n \geq 0$, a **Bott tower of height n**

$$X_n \longrightarrow X_{n-1} \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \{\text{point}\}$$

is defined inductively as an iterated \mathbb{P}^1 -bundle so that

$$X_k = \mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L}_{k-1})$$

for a line bundle \mathcal{L}_{k-1} over X_{k-1} .

So $X_1 = \mathbb{P}^1$ and X_2 is a Hirzebruch surface and so on.

Fan structure of a Bott tower

- Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus with character lattice $M := \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n$ and the cocharacter lattice $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.
- Let Δ_n be an n -dimensional nonsingular complete fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ which defines the toric variety X_n under the action of the torus T . the edges are

$$\begin{aligned}v_1 &= e_1, \dots, v_n = e_n, \\v_{n+1} &= -e_1 + c_{1,2}e_2 + \dots + c_{1,n}e_n, \\&\vdots \\v_{n+i} &= -e_i + c_{i,i+1}e_{i+1} + \dots + c_{i,n}e_n, 1 \leq i < n, \\v_{2n} &= -e_n.\end{aligned}\tag{0.1}$$

- The maximal cones are generated by these edges such that no cone contains both the edges v_i and v_{n+i} for $i = 1, \dots, n$.

Picard group of a Bott tower

- It follows that any k -th stage Bott tower arises from a collection of integers $\{c_{i,j}\}_{1 \leq i < j \leq n}$ as in (0.1). These integers are called the *Bott numbers* of the given Bott tower.
- We will restrict our attention to the case when the Bott numbers $\{c_{i,j}\}_{1 \leq i < j \leq n}$ are **all positive integers**.
- The Picard group of the Bott tower is

$$\mathrm{Pic}(X_n) = \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n,$$

where D_i denote the invariant prime divisor corresponding to the edge v_{n+i} .

Theorem 11 (Khan, _ (2019))

Let $D = \sum_{i=1}^k a_i D_i$ be a Cartier divisor on X_n . Then D is ample (respectively, nef) if and only if $a_i > 0$ (respectively, $a_i \geq 0$) for all $i = 1, \dots, n$.

Construction of a special class of subvarieties $X_i^{(j)}, 1 \leq j \leq i \leq n$

Fix a point $x \in X_n$. Set $X_i^{(1)} := X_i$ for every $1 \leq i \leq n$. For every $2 \leq i \leq n$, consider

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_i \xrightarrow{\pi_i} X_{i+1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\pi_n} X_1$$

Define $X_i^{(2)} := \pi_i^{-1}(\pi_n(x))$ for $i = 2, \dots, n$. Note that $x \in X_n^{(2)}$.

Then

$$X_n^{(2)} \longrightarrow X_{n-1}^{(2)} \longrightarrow \cdots \longrightarrow X_i^{(2)} \xrightarrow{\pi_{2,i}} X_{i+1}^{(2)} \longrightarrow \cdots \longrightarrow X_2^{(2)} \xrightarrow{\pi_{2,n}} X_1^{(2)}$$

is a

Bott tower.

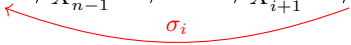
For every $3 \leq i \leq n$, define $X_i^{(3)} := \pi_{2,i}^{-1}(\pi_{2,n}(x))$.

$$\begin{array}{ccccccc}
 \mathbb{P}^1 = X_n^{(n)} & \subset & X_n^{(n-1)} & \subset & \dots & \subset & X_n^{(3)} & \subset & X_n^{(2)} & \subset & X_n^{(1)} \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{P}^1 = X_{n-1}^{(n-1)} & \subset & \dots & \subset & X_{n-1}^{(3)} & \subset & X_{n-1}^{(2)} & \subset & X_{n-1}^{(1)} \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \dots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \mathbb{P}^1 = X_3^{(3)} & \subset & X_3^{(2)} & \subset & X_3^{(1)} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & \mathbb{P}^1 = X_2^{(2)} & \subset & X_2^{(1)} \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \mathbb{P}^1 = X_1^{(1)}
 \end{array}$$

Proposition 12 (Biswas-__-Hanumanthu-Khan)

Each vertical tower is a Bott tower with positive Bott numbers.

Let us consider the composition of section maps

$$X_n^{(i)} \longrightarrow X_{n-1}^{(i)} \longrightarrow \cdots \longrightarrow X_{i+1}^{(i)} \longrightarrow X_i^{(i)}$$


for $1 \leq i \leq n$.

- Define $\Gamma_n^{(i)} := \sigma_i(X_i^{(i)}) \subset X_n^{(i)}$.
- We have $\Gamma_n^{(i)} \subset X_n^{(i)}$ for each i and $\Gamma_n^{(n)} = X_n^{(n)}$.
- We denote $\Gamma_n^{(1)}$ also by Γ_n .

Proposition 13 (Biswas-__-Hanumanthu-Khan)

The curves $\Gamma_n, \Gamma_n^{(2)}, \dots, \Gamma_n^{(n)}$ span $\overline{\text{NE}}(X_n)$, and they are dual to D_1, \dots, D_n .

Theorem 14 (Biswas-__-Hanumanthu-Khan)

The Seshadri constant of a nef line bundle \mathcal{L} on X_n at a point x is given as follows:

$$\varepsilon(X_n, \mathcal{L}, x) = \min_i \left\{ \mathcal{L} \cdot \Gamma_n^{(i)} \mid x \in \Gamma_n^{(i)} \right\}.$$

Corollary 15

Let $\mathcal{L} \equiv a_1 D_1 + \dots + a_n D_n$ be a nef line bundle on X_n .

- $\varepsilon(\mathcal{L}, 1) = a_n$.
- $\varepsilon(\mathcal{L}) = \min \{a_1, \dots, a_n\}$.

Theorem 16 (__ - Khan- Aditya)

Let \mathcal{E} be an equivariant nef vector bundle of rank r on X_3 satisfying “certain” conditions. Then the Seshadri constants of \mathcal{E} at any $x \in X_3$ are given by

$$\varepsilon(X_3, \mathcal{E}, x) = \min_i \left\{ \mu_{\min}(\mathcal{E}|_{\Gamma_3^{(i)}}) \mid x \in \Gamma_3^{(i)} \right\}.$$

Example 17

Let $\mathcal{L} \equiv D_1 + 3D_2 + 8D_3 + 4D_4 \in \text{Pic}(X_4)$ and $x \in X_4$. Then

$$\varepsilon(X_4, \mathcal{L}, x) = \begin{cases} 1, & \text{if } x \in \Gamma_4, \\ 3, & \text{if } x \notin \Gamma_4, x \in \Gamma_4^{(2)}, \\ 4, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(2)}. \end{cases}$$

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Thank You