The Stiefel–Whitney classes of a moment-angle manifold are trivial

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§1. Introduction

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Definition of a moment-angle manifold

Hereafter, K and m always denote a simplicial complex and the number of its vertices.
For $\sigma \in K$, put $Z_\sigma := \{(z_1, \dots, z_m) \in (D^2)^m | z_i \in S^1 \text{ if } i \notin \sigma\}$.

Definition (moment-angle complex)

For a simplicial complex K, the moment-angle complex Z_K for K is defined by

$$Z_K := \bigcup_{\sigma \in K} Z_\sigma.$$

In this talk, a **moment-angle manifold** means a moment-angle complex which is also a topological manifold without boundary, not necessarily smooth.

The simplest example of a moment-angle mfd is $Z_{\partial\Delta^d} = S^{2d-1}$.

Problem

Is any moment-angle manifold null-cobordant (i.e. the boundary of some manifold) ?

This problem is easily proven for smooth Z_K (and, as mentioned later, still open for not smooth cases).

Proof.

Consider the canonical $T^m = (S^1)^m$–action on $Z_K \subseteq (D^2)^m$ and let $T \subseteq T^m$ be the diagonal subtorus.

If Z_K is smooth, we obtain a principal S^1–bundle $Z_K \to Z_K/T$.

This is the sphere bundle for some complex line bundle, and therefore the total space Z_K is the boundary of its disk bundle. □

Goal of this talk

Moreover, we can prove the following theorem, which does not require the smoothness.

Theorem (H–Kishimoto–Kizu)

For a (real) moment-angle manifold, the Stiefel–Whiney classes vanish.

The definition of the Stiefel–Whitney classes for a topological manifold is explained in the next section.

This theorem includes the claim in the previous page, because it implies the vanishing of Stiefel–Whitney numbers. (Recall that a smooth manifold is null-cobordant iff its Stiefel–Whitney numbers vanish.)

However, this equivalence does not hold for topological manifolds, so the problem is still open for not smooth ones.

Ideas

The fundamental idea for this research is very simple.

(1) As is well-known, if Z_K is smooth, the SW classes of Z_K can be discribed as the images of equivariant ones, under the map $Z_K \to B_{T^m}Z_K = (Z_K \times ET^m)/T^m$.

(2) Let us consider the fiber sequence $Z_K \to B_{T^m}Z_K \to BT^m$.

It is also known that $B_{T^m}Z_K \simeq DJ_K$, the Davis–Januszkiewicz space, and $B_{T^m}Z_K \to BT^m$ is surjective in cohomology.

By considering the Leray–Serre spectral sequence, we see $Z_K \to B_{T^m}Z_K$ is trivial in cohomology.

In conclusion, the SW classes of Z_K are in the image of $H^*(B_{T^m}Z_K;\mathbb{Z}/2) \to H^*(Z_K : \mathbb{Z}/2)$ and this map is trivial, so the SW classes are equal to 0.

Then, to prove the theorem, it suffices to show that this argument also holds for topological moment-angle manifolds, namely, to define the equivariant SW classes for topological manifolds so that they are mapped to the ordinary ones.

(This argument also holds for real moment-angle manifold, but I will omit the details in order to avoid a repetition.)

§2. (Equivariant) Stiefel–Whitney classes for topological manifolds

- Stiefel–Whitney classes for topological manifolds
- Equivariant Stiefel–Whitney classes for topological manifolds

Stiefel–Whitney classes for topological manifolds

Let M be an n–dim closed connected topological manifold.

We can define the **topological tangent fiber space** $\mathcal{J}$ of M as follows so that it is equivalent to the tangent bundle if M is smooth.

- $\mathcal{J}_0 := \{\ell\colon [0,1] \to M \mid \ell(t) \neq \ell(0) \text{ for } \forall t \neq 0\}$.
- $\mathcal{J} := \mathcal{J}_0 \cup \{\ell\colon [0,1] \to M \mid \ell(t) = \ell(0) \text{ for } \forall t \neq 0\}$.
- $p\colon \mathcal{J} \to M$ defined by $p(\ell) = \ell(0)$.

Theorem (Fadell 1965)

If M is orientable over a ring R, there is $\Phi \in H^n(\mathcal{J}, \mathcal{J}_0; R)$ such that $\phi\colon H^*(M;R) \to H^{*+n}(\mathcal{J}, \mathcal{J}_0; R)$ defined by $\phi(u) = p^*(u) \cup \Phi$ is an isomorphism.

Then, in the same way for smooth manifolds, we can define the i–th SW class of M by $w_i(M) = \phi^{-1}(\mathrm{Sq}^i \phi(1))$.

This page is a note on the Thom class.

Theorem (Fadell 1965)

If M is orientable over a ring R, there is $\Phi \in H^n(\mathcal{J}, \mathcal{J}_0; R)$ such that $\phi\colon H^*(M; R) \to H^{*+n}(\mathcal{J}, \mathcal{J}_0; R)$ defined by $\phi(u) = p^*(u) \cup \Phi$ is isomorphism.

More precisely, the Thom class Φ is characterized by the condition that, for each $x \in M$, Φ restricts to a generator of $H^n(\mathcal{J}|_x, \mathcal{J}_0|_x; R) \cong R$.

Recall that an orientation of a topological manifold M is a collection of generators $U_x \in H^n(M, M \setminus \{x\}; R)$ satisfying certain pasting condition.

Since p restricts to $H^n(M, M \setminus \{x\}; R) \cong H^n(\mathcal{J}|_x, \mathcal{J}_0|_x; R)$, if we choose an orientation of M, the Thom class Φ is uniquely determined.

Equivariant SW classes for topological manifolds

Suppose that a topological group G acts on M.

If we define a G–action on $\mathcal{J}$ by $(g \cdot \ell)(t) = g \cdot \ell(t)$, then $p \colon \mathcal{J} \to M$ is G–equivariant.

Then we obtain the following from $G \to EG \to BG$.

- $M \to B_G M \to BG$
- $(\mathcal{J}, \mathcal{J}_0) \to (B_G \mathcal{J}, B_G \mathcal{J}_0) \to BG$

We assume that these fiber sequences are R–orientable, that is, the cohomology local coefficient systems over R are trivial.

The following is immediate from the Leray–Serre spectral sequence of $(\mathcal{J}, \mathcal{J}_0) \to (B_G \mathcal{J}, B_G \mathcal{J}_0) \to BG$.

Proposition

There is a unique $\Phi_G \in H^n(B_G \mathcal{J}, B_G \mathcal{J}_0; R)$ which restricts to Φ.

Let E_r be the Leray–Serre spectral sequence of $M \to B_G M \to BG$ and $\hat{E}_r$ be that of $(\mathcal{J}, \mathcal{J}_0) \to (B_G\mathcal{J}, B_G\mathcal{J}_0) \to BG$.

Additionally, let $p_G \colon B_G\mathcal{J} \to B_G M$ be the map induced from $p \times \mathrm{id}_{EG}$.

Theorem (H–Kishimoto–Kizu)

$\phi_G \colon H^*(B_G M; R) \to H^{*+n}(B_G\mathcal{J}, B_G\mathcal{J}_0; R)$ defined by $\phi_G(u) = p_G^*(u) \cup \Phi_G$ is an isomorphism.

To explain the proof shortly, if we take $\Psi_G \in C^n(B_G\mathcal{J}, B_G\mathcal{J}_0; R)$ representing Φ_G and define a map $\psi_G \colon C^*(B_G M; R) \to C^*(B_G\mathcal{J}, B_G\mathcal{J}_0; R)$ by $\psi_G(u) = p_G^*(u) \cup \Psi_G$, then it respects the filtrations of Leray–Serre spectral sequence.

Thus we see that ψ_G induces a homomorphism $E_1^{*,*} \to \hat{E}_1^{*,*+n}$ and it is an isomorphism in E_2–page by construction.

Now we can define the equivariant SW classes of a topological manifold M in the same way as ordinary ones.

Definition

The i–th equivariant SW class of M is $w_i^G(M) = \phi_G^{-1}(\mathrm{Sq}^i \phi_G(1))$.

Remark

As is almost clear from the construction, $w_i^G(M) \mapsto w_i(M)$ under $M \to B_G M$.

Thus we see that the vanishing of SW classes of topological moment-angle manifold can be proven in the same way as smooth ones.

§3. Further problem

Recall that this research starts from the problem whether a moment-angle manifold is null-cobordant or not, and to solve it affirmatively, we use the S^1–bundle $Z_K \to Z_K/T$ where T is a 1–dim subtorus freely acting on Z_K.

More generally, let us consider the class of manifolds which are described as Z_K/T where T is a subtorus of T^m acting on Z_K freely.

For this class, the vanishing of SW classes seems to occur accidentally. (Probably we can share this intuition if you consider this problem for quasitoric manifolds.)

On the other hand, the vanishing of SW numbers seems to hold in many cases, and the following problem seems still worth considering.

Problem

Let K be an $(n-1)$–dim simplicial complex with m vertices and $T \subseteq T^m$ act on Z_K freely.

If Z_K/T is a manifold and dim $T < m - n$, do the SW numbers of Z_K/T vanish?

If there is T' satisfying the same conditions as T and $T \subsetneq T'$, then this problem is easily solved affirmatively since the Borel construction map $Z_K/T \to B_{T^m/T}Z_K/T \simeq DJ_K$ factors through $Z_K/T' \to B_{T^m/T'}Z_K/T' \simeq DJ_K$. (Due to the difference of dimension, $Z_K/T \to Z_K/T'$ is 0 in the top cohomology group.)