Almost complex torus manifolds - graphs, Hirzebruch genera, and problem of Petrie type

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March 2022

2022 Toric Topology

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This talk is based on 'Almost complex torus manifolds - graphs, Hirzebruch genera, and problem of Petrie type', arXiv:2201.00352.

An **almost complex manifold** is a manifold M with a bundle map $J : TM \to TM$, called an **almost complex structure**, which restricts to a linear complex structure on each tangent space. ($J_m^2 = -\mathrm{Id}_{T_m M}$ for all $m \in M$.)

Throughout this talk, any manifold is compact and almost complex, and any group action on an almost complex manifold is assumed to preserve the almost complex structure, that is, $dg \circ J = J \circ dg$ for all elements g of the group.

Let a k-dimensional torus T^k act on M. Let F be a fixed component of the action. Let $\dim F = 2m$ and let p be a point in F. The normal space $N_p F$ of N at p decomposes into the sum of $n-m$ complex 1-dimensional vector spaces $L_1, \cdots, L_{n-m}$, where on each L_i the torus T^k acts by multiplication by $g^{w_{p,i}}$ for all $g \in T^k$, for some non-zero element $w_{p,i}$ of $\mathbb{Z}^k$, $1 \leq i \leq n-m$. These elements $w_{p,1}, \cdots, w_{p,n-m}$ are the same for all $p \in F$ and called **weights** of F.

Rough idea: getting a multigraph

Let a torus T^k act on M with isolated fixed points.

To each fixed point p we assign a vertex, also denoted by p.

Let w be a weight at a fixed point p.

Then there is another fixed point q that has weight $-w$. (p and q are in the same isotropy submanifold fixed by an action of a subgroup, whose elements fix w.)

We draw an edge from p to q, giving a label w to the edge.

This way we get a directed labeled multigraph.

Definition

A **labeled directed** k**-multigraph** Γ is a set V of vertices, a set E of edges, maps $i : E \to V$ and $t : E \to V$ giving the initial and terminal vertices of each edge, and a map $w : E \to \mathbb{Z}^k$ ($\mathbb{N}^+$ if $k = 1$) giving the label of each edge.

Let $w = (w_1, \cdots, w_k)$ be an element in $\mathbb{Z}^k$. By $\ker w$ we shall mean the subgroup of T^k whose elements fix w. That is,

$$\ker w = \{g = (g_1, \cdots, g_k) \in T^k \subset \mathbb{C}^k \,|\, g^w := g_1^{w_1} \cdots g_k^{w_k} = 1\}.$$

Definition

Let a k-dimensional torus T^k act on a compact almost complex manifold M with isolated fixed points. We say that a (labeled directed k-)multigraph $\Gamma = (V, E)$ **describes** M if the following hold:

(i) The vertex set V is equal to the fixed point set M^{T^k}.

(ii) The multiset of the weights at p is $\{w(e) \,|\, i(e) = p\} \cup \{-w(e) \,|\, t(e) = p\}$ for all $p \in M^{T^k}$.

(iii) For each edge e, the two endpoints $i(e)$ and $t(e)$ are in the same component of the isotropy submanifold $M^{\ker w(e)}$.

$\ker w$ is a subgroup of T^k and the set $M^{\ker w}$ of points in M that are fixed by the $\ker w$-action is a union of smaller dimensional compact almost complex submanifolds of M.

Proposition (J)

Let a k-dimensional torus T^k act on a compact almost complex manifold M with isolated fixed points. There exists a (labeled directed k-)multigraph Γ describing M that has no self-loops.

Idea of proof - This was proved when $k = 1$ by Jang-Tolman.

For $k > 1$: take a weight w at some fixed point p, consider a fixed component F of $M^{\ker w}$, find a suitable subcircle S^1 of T^k, apply the above result for S^1-action on F, and replace the S^1-weight by T^k-weight.

Let T^2 act on $\mathbb{CP}^3$ by

$$(g_1, g_2) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g_1 z_1 : g_1^2 z_2 : g_2 z_3]$$

for all $g = (g_1, g_2) \in T^2 \subset \mathbb{C}^2$. The action has 4 fixed points, $p_0 = [1 : 0 : 0 : 0]$, $p_1 = [0 : 1 : 0 : 0]$, $p_2 = [0 : 0 : 1 : 0]$, and $p_3 = [0 : 0 : 0 : 1]$.
Near p_2, using local coordinates $(\frac{z_0}{z_2}, \frac{z_1}{z_2}, \frac{z_3}{z_2})$, T^2 acts near p_2 by

$$(g_1, g_2) \cdot (\frac{z_0}{z_2}, \frac{z_1}{z_2}, \frac{z_3}{z_2}) = (\frac{z_0}{g_1^2 z_2}, \frac{g_1 z_1}{g_1^2 z_2}, \frac{g_2 z_3}{g_1^2 z_2}) = (g_1^{-2} \frac{z_0}{z_2}, g_1^{-1} \frac{z_1}{z_2}, g_1^{-2} g_2 \frac{z_3}{z_2})$$

and hence the weights at p_2 are $\{(-2, 0), (-1, 0), (-2, 1)\}$. The weights at the fixed points p_0, p_1, and p_3 are $\{(1, 0), (2, 0), (0, 1)\}$, $\{(-1, 0), (1, 0), (-1, 1)\}$, and $\{(0, -1), (1, -1), (2, -1)\}$, respectively.
Note: This action is not GKM; the weights $\{(1, 0), (2, 0), (0, 1)\}$ at p_0 are not pairwise linearly independent, but we can still associate a multigraph.

$\{(1,0),(2,0),(0,1)\}$, $\{(-1,0),(1,0),(-1,1)\}$,
$\{(-2,0),(-1,0),(-2,1)\}$, $\{(0,-1),(1,-1),(2,-1)\}$

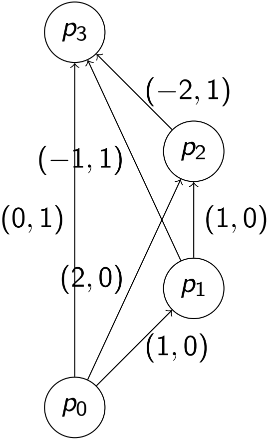

Figure: Multigraph describing $\mathbb{CP}^3$

Ahara explicitly described circle actions on the Fano 3-folds V_5 and V_{22} with 4 fixed points that have weights

$$\{1,2,3\},\ \{-1,1,a\},\ \{-1,-a,1\},\ \{-1,-2,-3\},$$

where $a = 4$ for V_5 and $a = 5$ for V_{22}. McDuff provided compatible symplectic structures on these actions on V_5 and V_{22}.

$$\{1,2,3\},\ \{-1,1,a\},\ \{-1,-a,1\},\ \{-1,-2,-3\}$$

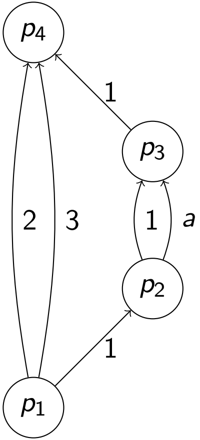

Figure: Multigraph describing Fano 3-fold

Definition

An **almost complex torus manifold** is a $2n$-dimensional compact almost complex manifold equipped an effective T^n-action that has fixed points.

By definition, an almost complex torus manifold has only isolated fixed points.

Theorem (J)

For an almost complex torus manifold, a multigraph describing it has no multiple edges.

Idea of proof - Suppose there are two edges between p and q with labels $w_{p,1}, w_{p,2}$. Because p and q are in the same component of $M^{\ker w_{p,1}}$ and $M^{\ker w_{p,2}}$, the weights $w_{p,1}, \cdots, w_{p,n}$ at p and the weights $w_{q,1}(= -w_{p,1}), w_{q,2}(= -w_{p,2}), w_{q,3}, \cdots, w_{q,n}$ at q are equal modulo $w_{p,1}$ and $w_{p,2}$. With the fact that $w_{q,1}, \cdots, w_{q,n}$ span $\mathbb{Z}^n$, this cannot happen.

For a compact almost complex manifold, the **Hirzebruch χ_y-genus** is the genus of the power series $\frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}}$, and the **Todd genus** is the genus of the power series $\frac{x}{1-e^{-x}}$.

The power series for the Todd genus is obtained by taking $y = 0$ in the power series for the Hirzebruch χ_y-genus.

Denote by $\chi_y(M)$ the Hirzebruch χ_y-genus of M.

The Hirzebruch χ_y-genus of M contains three topological information; the Euler number (when $y = -1$), the Todd genus (when $y = 0$), and the signature (when $y = 1$) of M.

Let M be a $2n$-dimensional compact almost complex manifold.

Let $\chi_y(M) = \sum_{i=0}^{n} a_i(M) \cdot (-y)^i$ denote the Hirzebruch χ_y-genus $\chi_y(M)$ of M, for some integers $a_i(M)$, $0 \leq i \leq n$.

Note that the standard convention is $\chi_y(M) = \sum_{i=0}^{n} \chi^i(M) \cdot y^i$; hence $a_i(M) = (-1)^i \chi^i(M)$.

We shall use the coefficients $a_i(M)$ so that statements and proofs become clearer. In this convention, if a torus acts on M with isolated fixed points, each fixed point contributes some $a_i(M)$ by $+1$.

If we use $\chi^i(M)$ then there is a sign issue that each isolated fixed point contributes $\chi^i(M)$ by $(-1)^i$ for some i.

Theorem (Kosniowski formula)

Let the circle act on a compact unitary manifold M. For each connected component F of the fixed point set M^{S^1}, let $d(-,F)$ and $d(+,F)$ be the numbers of negative weights and positive weights in the normal bundle NF of F, respectively. Then the Hirzebruch χ_y-genus $\chi_y(M)$ of M satisfies

$$\chi_y(M) = \sum_{F \subset M^{S^1}} (-y)^{d(-,F)} \cdot \chi_y(F) = \sum_{F \subset M^{S^1}} (-y)^{d(+,F)} \cdot \chi_y(F).$$

Remark:
By the formula, $a_i(M) = a_{n-i}(M)$ for all i.
If the action on M has isolated fixed points, then $a_i(M)$ is equal to the number of fixed points that have exactly i negative weights.

Theorem (J)

Let M be a $2n$-dimensional almost complex torus manifold. Then $a_i(M) > 0$ for $0 \leq i \leq n$, where $\chi_y(M) = \sum_{i=0}^{n} a_i(M) \cdot (-y)^i$ is the Hirzebruch χ_y-genus of M. In particular, the Todd genus of M is positive.

Idea of proof - by induction and the Kosniowski formula
Consider a characteristic submanifold F_0, a real codimension 2 submanifold fixed by a subcircle S^1 of T^n.
There is a quotient action $T^{n-1} = T^n/S^1$ on F_0. By inductive hypothesis all coefficients of the Hirzebruch χ_y-genus of F_0 are positive; $a_i(F_0) > 0$ for $0 \leq i \leq n-1$.
By the Kosniowski formula

$$\chi_y(M) = \sum_{F \subset M^{S^1}} (-y)^{d(-,F)} \cdot \chi_y(F) = \sum_{F \subset M^{S^1}} (-y)^{d(+,F)} \cdot \chi_y(F)$$

the two equations imply that $a_i(M) > 0$ for $0 \leq i \leq n$.

We will use this later.

Corollary

Let M be a $2n$-dimensional almost complex torus manifold with Euler number $n+1$. Then M has exactly $n+1$ fixed points and $a_i(M) = 1$ for $0 \leq i \leq n$.

Proof.

The Euler number of M is equal to the sum of Euler numbers of its fixed points. Since there are only isolated fixed points and the Euler number of a point is 1, there are exactly $n+1$ fixed points. Each fixed point contributes some $a_i(M)$ by 1.
The Hirzebruch χ_y-genus of M with $y = -1$ is the Euler number of M, which is $n + 1 = \sum_{i=0}^{n} a_i(M)$. By the theorem in the previous slide $a_i(M) > 0$ and hence $a_i(M) = 1$ for all i. $\square$

Problem of Petrie type

Petrie conjectured that if a homotopy $\mathbb{CP}^n$ admits a non-trivial S^1-action, then it has the same Pontryagin class as $\mathbb{CP}^n$.

Conjecture (Petrie's conjecture)

Let M be a $2n$-dimensional compact oriented manifold which is homotopy equivalent to the complex projective space $\mathbb{CP}^n$. If M admits a non-trivial S^1-action, the total Pontryagin class of M agrees with that of $\mathbb{CP}^n$.

A weak version of the conjecture adds an assumption that the action on M has isolated fixed points.

Conjecture (Petrie's conjecture)

If a homotopy $\mathbb{CP}^n$ admits a non-trivial S^1-action, its total Pontryagin class agrees with that of $\mathbb{CP}^n$.

While the Petrie's conjecture remains open in its full generality, there are many partial or related results.

Petrie proved the conjecture if a homotopy $\mathbb{CP}^n$ admits an action of the torus T^n instead of an S^1-action.

Dejter confirmed the Petrie's conjecture in dimension up to 6.

Musin confirmed it in dimension up to 8 (weak version).

James confirmed in dimension 8.

Dessai and Wilking reduced the dimension of a torus acting on a manifold by showing that the conclusion of the Petrie's conjecture holds if a homotopy $\mathbb{CP}^n$ admits a T^k-action with $2n \leq 8k - 4$.

The Petrie's conjecture can be thought of as a particular case of the following question.

Question

If a manifold equipped with a group action (torus action) shares some information with $\mathbb{CP}^n$, then what other information does the manifold share with $\mathbb{CP}^n$?

Tsukada and Washiyama and Masuda proved that the conclusion of the Petrie's conjecture holds if a compact oriented S^1-manifold with the same cohomology as $\mathbb{CP}^n$ has three or four fixed components, respectively.

Hattori proved the Petrie's conjecture under an assumption that a compact unitary manifold has the same cohomology of $\mathbb{CP}^n$ with first Chern class $(n+1)x$ and admits an S^1-action.

Tolman asked a symplectic analogue of the Petrie's conjecture that, if a compact symplectic manifold M with $H^{2i}(M;\mathbb{R}) \cong H^{2i}(\mathbb{CP}^n;\mathbb{R})$ for all i admits a Hamiltonian S^1-action, then $H^j(M;\mathbb{Z}) \cong H^j(\mathbb{CP}^n;\mathbb{Z})$ for all j. In the same paper Tolman answered this question affirmatively in dimension up to 6, and the work of Godinho and Sabatini together with the work of Tolman and the author confirmed it in dimension 8.

Motivated by the Petrie's conjecture, Masuda and Suh asked if an isomorphism between the cohomology rings of two torus manifolds preserves the Pontryagin classes of them.

For an almost complex torus manifold M, having the same Euler number as $\mathbb{CP}^n$ is enough to force M to share many other invariants with $\mathbb{CP}^n$.

Theorem (J)

Let M be a $2n$-dimensional almost complex torus manifold with Euler number $n+1$. Then the following invariants of M and $\mathbb{CP}^n$ are equal. Here, $\mathbb{CP}^n$ is equipped with a linear T^n-action.

1. *A (multi)graph describing it.*
2. *The weights at the fixed points.*
3. *All the Chern numbers.*
4. *Equivariant cobordism class.*
5. *The Hirzebruch χ_y-genus.*
6. *The Todd genus.*
7. *The signature.*

If furthermore the action on M is equivariantly formal, the following invariants of M and $\mathbb{CP}^n$ are also equal.

(8) *The rational equivariant cohomology.*

(9) *The Chern classes.*

- (1) implies (2) by definition.
- (2) implies (3) because Chern numbers are computed in terms of the weights at the fixed points in the Atiyah-Bott-Berline-Vergne localization formula.
- (3) implies (4) because two manifolds are equivariantly cobordant if and only if they have the same Chern numbers.
- (3) implies (5) because the coefficients of the Hirzebruch χ_y-genus can be computed as rational combinations of the Chern numbers.
- (5) implies (6) and (7).
- Alternatively, having the Euler number $n+1$ implies (5).
- (1) implies (8) if in addition the action on M is equivariantly formal, because our graph is a GKM graph, and a GKM graph determines the equivariantly cohomology of a given manifold.
- (9) follows from (1) and Proposition 3.4 of Goertsches, Konstantis, and Zoller. Proposition 3.4 states that our graph describing an almost complex torus manifold (which is called a signed graph in their paper) determines its Chern classes if it is equivariantly formal.

By a linear T^n-action on $\mathbb{CP}^n$ we mean an action

$$g \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : g^{a_1} z_1 : \cdots : g^{a_n} z_n]$$

for all $g \in T^n \subset \mathbb{C}^n$, where $a_1, \cdots, a_n$ form a basis of $\mathbb{Z}^n$. Let $a_0 = (0, \cdots, 0) \in \mathbb{Z}^n$. The action has $n+1$ fixed points $p_0 = [1 : 0 : \cdots : 0], \cdots, p_n = [0 : \cdots : 0 : 1]$, and the weights at p_i are $\{a_j - a_i\}_{j \neq i}$.

Now, we associate a (multi)graph describing this T^n-action on $\mathbb{CP}^n$. To each fixed point p_i we assign a vertex (also denoted by p_i). For $i < j$, the fixed points p_i and p_j are in the fixed component $[0 : \cdots : 0 : z_i : 0 : \cdots : 0 : z_j : 0 : \cdots : 0]$ of $M^{\ker(a_j - a_i)}$, which is the 2-sphere, on which the T^n-action on $\mathbb{CP}^n$ restricts to act by

$$\begin{aligned} & g \cdot [0 : \cdots 0 : z_i : 0 : \cdots : 0 : z_j : 0 : \cdots : 0] \\ = & [0 : \cdots : 0 : g^{a_i} z_i : 0 : \cdots : 0 : g^{a_j} z_j : 0 : \cdots : 0], \end{aligned}$$

giving p_i and p_j weight $a_j - a_i$ and $a_i - a_j$ for this action, respectively. Therefore, for $i < j$ we draw an edge from p_i to p_j and give the edge label $a_j - a_i$. Let Γ be a (multi)graph obtained. Then Γ describes the linear T^n-action on $\mathbb{CP}^n$.

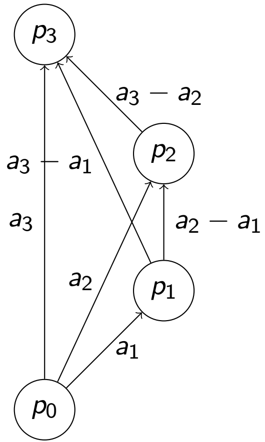

Figure: Graph describing a linear T^3-action on $\mathbb{CP}^3$

Since $\chi(M) = n+1$, there are only finitely many fixed points, and the Euler number of a point is 1, M has exactly $n+1$ fixed points.

By the proposition, there is a multigraph Γ describing M that has no multiple edges and no self-loops.

We label the $n+1$ fixed points (vertices) by p_0, p_1, $\cdots$, p_n.

Since there are $n+1$ vertices, each vertex has n edges, and there are no multiple edges between any two vertices, for $i \neq j$, there is exactly one edge between p_i and p_j.

For $0 < i < j$, we may assume that an edge $e_{i,j}$ between p_i and p_j is from p_i to p_j (by reversing the direction and the label). Let $w_{i,j}$ be the label of $e_{i,j}$.

We will show that $w_{i,j} = w_{0,j} - w_{0,i}$ for $0 < i < j$, thus the graph is the same as one describing the linear action on $\mathbb{CP}^n$.

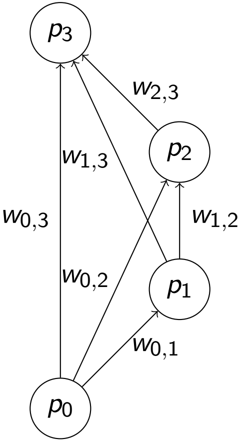

Figure: Graph describing M

We need to show if $0 < i < j$, then $w_{i,j} = w_{0,j} - w_{0,i}$, so that this graph agrees with a linear T^n-action on $\mathbb{CP}^n$.

Fix $0 < i < j$. Let $T = \ker w_{0,i} \cap \ker w_{0,j}$.

Since the weights $w_{0,1}, \cdots, w_{0,n}$ at p_0 form a basis of $\mathbb{Z}^n$, T is an $(n-2)$-dimensional subtorus of T^n.

Let F_0 be a connected component of M^T which contains p_0.

Since T only fixes $w_{0,i}$ and $w_{0,j}$ among the weights at p_0, it follows that F_0 is 4-dimensional and F_0 contains p_0, p_i, and p_j.

If F_0 contains another fixed point, then $a_0(F_0) + a_1(F_0) + a_2(F_0) \geq 4$ with $a_i(F_0) > 0$ and hence some $a_i(F_0)$ must be at least 2.

However, applying the Kosniowski formula for a suitable subcircle of T^n we have

$$\chi_y(M) = \sum_{F \subset M^S} (-y)^{d(-,F)} \cdot \chi_y(F),$$

where for each fixed component $F \subset M^S = M^T$, $d(-,F)$ is the number of negative S-weights of F.

In $\chi_y(M) = \sum_{i=0}^n a_i(M) \cdot (-y)^i$, the above implies that some $a_i(M)$ must be at least 2, but we know $a_i(M) = 1$ for all i. Hence F_0 only contains p_0, p_i, and p_j.

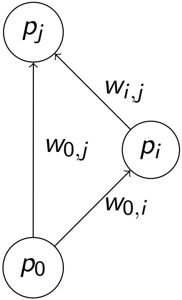

Figure: Graph describing T^n-action on F_0

Finally, for the T^n-action on F_0, we push-forward the equivariant cohomology class 1 in the ABBV localization formula. It follows that $w_{i,j} = w_{0,j} - w_{0,i}$.

$$0 = \int_{F_0} 1 = \sum_{p \in F_0^{T^n}} \frac{1}{e_{T^n}(N_p F_0)} = \sum_{p \in F_0^{T^n}} \frac{1}{e_{T^n}(T_p F_0)} = \frac{1}{w_{0,i} w_{0,j}} + \frac{1}{(-w_{0,i}) w_{i,j}} + \frac{1}{(-w_{0,j})(-w_{i,j})}.$$

Thus, the graph describing M agrees with the graph describing a linear T^n-action on $\mathbb{CP}^n$.

Thank you very much for your attention :)