Iterated residue formula of generalized Bott manifolds and its applications

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Toric Topology 2022 in Osaka

Osaka City University (online), March 25, 2022

1. Motivation
2. Brief introduction to generalized Bott manifolds
3. Iterated residue formula
4. Applications

Common objects in geometric (topological) study

- Product of projective spaces. e.g. $\mathbb{C}P^n \times \mathbb{C}P^m$;
- Complete intersections in product of projective spaces. e.g. Milnor hypersurfaces H_{n_1,n_2} Poincaré dual to $u+v$ in $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$;
- Homogeneous spaces of compact Lie groups. e.g. structure groups, Grassmannian manifolds;
- Principle G bundle, where G is a compact Lie group. e.g. $U(n)$ bundles;
- Toric varieties which are 1-1 corresponding to rational fans.

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- Toric varieties which are 1-1 corresponding to rational fans.

Feature: They admit clear topology——tangential bundles and cohomology rings.
Furthermore, they also admit nice properties: such as complex structure, symplectic structure, group actions, Riemannian metric of positive (or non negative) sectional (or Ricci, scalar) curvature...

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Quasi-toric manifolds

Let (M^{2n}, Λ) be a quasi-toric manifold over simple polytope P^n with m facets. Then M^{2n} admits stable complex structure

$$TM \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m$$

where each ρ_i is a complex line bundle corresponding to each facet.

$$H^*(M, \mathbb{Z}) \cong \mathbb{Z}[v_1, \cdots, v_m]/(I + J)$$

where $v_i = c_1(\rho_i)$, idea I comes from missing faces of P^n, idea J comes from characteristic matrix Λ.

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- Moment-angle manifolds and quasi-toric manifolds admit a Riemannian metric of positive scalar curvature;
- Moment-angle manifolds admit trivialized normal bundle, thus potentially can serve as representatives in framed bordism group (stable homotopy groups).

Index theory and rigidity

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Definition

Let R be an integral domain over $\mathbb{Q}$. Then a **genus** φ is a ring homomorphism $\varphi : \Omega^{SO} \otimes \mathbb{Q} \longrightarrow R$.
Every genus corresponds to a power series in R.

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Common index

L genus (signature), Todd genus, $\widehat{A}$ genus, Elliptic genus, Witten genus, Euler characteristic, Pontryagin numbers...

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A simply connected, closed and dim ≥ 5 spin manifold carries a Riemannian metric of positive scalar curvature if and only if its α invariant vanishes, where α invariant is a refined version of $\widehat{A}$ genus.

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If a compact string manifold M^{4k} carries a Riemannian metric of positive Ricci curvature, then the Witten genus $\varphi_W(M)$ vanishes.

Geometers are looking for simply connected manifolds that admits a metric of positive scalar curvature but no metric of positive Ricci curvature.

Rigidity of group action

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Topologists are interested in manifolds that admits non-trivial S^1 action but no non-trivial S^3 action.

Interesting problems

Since the kernel of ring homomorphism $\varphi_W : \Omega^{String} \longrightarrow MF$ precisely consists of (bordism classes of) $\mathbb{O}P^2$ (Cayley plane) bundles with connected structure group.

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Can every string cobordism class with vanishing Witten genus be represented by some manifolds with positive Ricci curvature?

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Hirzebruch prize question

Does there exist a 24-dimensional compact, differential string manifolds X with $\widehat{A}(X) = 1$ and $\widehat{A}(X, T_{\mathbb{C}}) = 0$?

Such manifold admits action of Monster by diffeomorphism. Mohowald-Hopkins constructed a 24-dimensional manifold from homotopy theory, but it is hard to know its geometry.

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In summary, one can study the geometric and topologic properties of quasi-toric manifolds by calculating all kinds of index and characteristic numbers.

In our work, we will focus on the calculation process and give an explicit formula of genus of generalized Bott manifolds.

Generalised Bott manifolds

Definition

A **generalised Bott tower** of height n is a tower of projective bundles

$$B_n \xrightarrow{p_n} B_{n-1} \xrightarrow{p_{n-1}} \cdots \longrightarrow B_2 \xrightarrow{p_2} B_1 \longrightarrow \mathrm{pt}$$

of complex manifolds, where $B_1 = \mathbb{C}P^{n_1}$ and each B_k is the complex projectivisation of a sum of n_k complex line bundles and one trivial line bundle over B_{k-1}.
The last stage B_n in a generalised Bott tower is called **generalised Bott manifolds**.

Tangential bundle

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Suppose the formal Chern roots of ξ_k are $\{x_{k1}, x_{k2}, \cdots, x_{kn_k}\}$, let $-u_k$ be the first Chern class of tautological line bundle η_k over $B_k = \mathbb{C}P(\xi_k \oplus \underline{\mathbb{C}})$.

Cohomology ring

$$H^*(B_n) \cong \mathbb{Z}[u_1, \cdots, u_n]/\langle f_i(u_1, \cdots, u_n) : \ i = 1, \cdots, n\rangle$$

where $f_i(u_1, \cdots, u_n) = u_i \cdot \prod_{j=1}^{n_i}(u_i + x_{ij})$.

Let X be the submanifold of B_n Poincaré dual to $x \in H^2(B_n;\mathbb{Z})$. Suppose ν denote the normal bundle of inclusion $i : X \hookrightarrow B_n$. Then $c_1(\nu) = i^*(x)$, $i^*(TB_n) \cong TX \oplus \nu$. Then

$$c(X) \cdot c(\nu) = i^*(c(B_n)); \quad p(X) \cdot p(\nu) = i^*(p(B_n))$$

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Complete intersection

Let $x_1, \cdots, x_r \in H^2(B_n;\mathbb{Z})$, The transversal intersection of $X_1, \cdots, X_r$ is called **complete intersections** of codimension $2r$.

For any genus ψ with the charateristic power series $Q(x) = x/f(x)$, we have

$$\psi(X) = \langle (\frac{u_1}{f(u_1)})^{n_1+1} \cdot f(x) \cdot \prod_{i=1}^{n} \frac{u_i}{f(u_i)} \prod_{j=1}^{n_i} \frac{u_i + x_{ij}}{f(u_i + x_{ij})}, [B_n] \rangle.$$

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The calculation of genus ψ is transferred into the calculation of degree dim B_n items in complex analysis.
If B_n is direct products of projective spaces, $\psi(X)$ can be simplified into a very neat expression. As to general B_n, the idea f_i's in $H^*(B_n; \mathbb{Z})$ is the biggest obstacle in calculating genus, the variables u_i's are dependent, we have to cope with the relation carefully.

iterated residue formula

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Theorem

For any $F \in H^{n_1+\cdots+n_n}(B_n;\mathbb{Z})$, we have

$$\langle\ F,\ [B_n]\ \rangle = \mathrm{Res}_0 \left\{ \mathrm{Res}_0 \cdots \left\{ \mathrm{Res}_0 \frac{F}{\prod_{j=1}^{n_i} f_i(u_1,\cdots,u_n)} du_1 \right\} \cdots du_{n-1} \right\} du_n$$

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Remark

It should be pointed out that the order of u_i's in the "iterated residue" can't be exchanged, otherwise we will get the result zero. This illustrates why global residue theorem fails.

Main theorem

Let X be the submanifold of generalised Bott manifold B_n Poincaré dual to $x \in H^2(B_n;\mathbb{Z})$.
For any genus ψ with the charateristic power series $Q(x) = x/f(x)$, we have

$$\psi(X) = \mathrm{Res}_0 \left\{ \cdots \left\{ \mathrm{Res}_0 \frac{f(x)}{f(u_1)^{n_1+1}} \cdot \prod_{i=1}^{n} \frac{1}{f(u_i)} \prod_{j=1}^{n_i} \frac{1}{f(u_i+x_{ij})} du_1 \right\} \cdots \right\} du_n.$$

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Remark

It is a pity that above result can not be generalized to quasi-toric manifolds. For arbitrary quasi-toric manifolds, the product of its Chern roots can not kill the ideas in $H^*(M;\mathbb{Z})$, thus $\psi(X)$ contains both function f and monomials in u_i.

Reformulation of α invariant

With the help of "iterated residue", we can simplify the computation of α invariant significantly.

$$
\begin{aligned}
&F_{n_1,n_2,\mathrm{I}}(d_1,d_2)\\
=&(\frac{1}{2\pi i})^2\oint_{\gamma_1}\left\{\oint_{\gamma_2}(\frac{1}{e^{u/2}-e^{-u/2}})^{n_1+1}\cdot\frac{e^{(d_1u+d_2v)/2}}{e^{v/2}-e^{-v/2}}\prod_{j=1}^{n_2}\frac{1}{e^{(v-i_ju)/2}-e^{-(v-i_ju)/2}}du\right\}dv\\
=&(\frac{1}{2\pi i})^2\oint_{\gamma_1}\left\{\oint_{\gamma_2}(\frac{1}{e^u-1})^{n_1+1}\cdot\frac{e^{\frac{(n_1+1+d_1+\sum i_j)u+(-n_2+1+d_2)v}{2}}}{e^v-1}\prod_{j=1}^{n_2}\frac{1}{1-e^{i_ju-v}}du\right\}dv
\end{aligned}
$$

let $t=e^u-1,\ s=e^v-1,\ k_1=\dfrac{n_1-1+d_1+\sum i_j}{2},\ k_2=\dfrac{-n_2-1+d_2}{2}$.

$$
\begin{aligned}
=&(\frac{1}{2\pi i})^2\oint_{\gamma_1}\frac{(1+s)^{k_2}}{s}\left\{\oint_{\gamma_2}\frac{(t+1)^{k_1}}{t^{n_1+1}}\cdot\prod_{j=1}^{n_2}\frac{1}{1-(1+t)^{i_j}/(1+s)}dt\right\}ds\\
=&(\frac{1}{2\pi i})^2\oint_{\gamma_1}\frac{(1+s)^{k_2}}{s}\left\{\oint_{\gamma_2}\frac{(t+1)^{k_1}}{t^{n_1+1}}\cdot\prod_{j=1}^{n_2}\frac{(1+s)/s}{1-\{(1+t)^{i_j}-1\}/s}dt\right\}ds\\
=&(\frac{1}{2\pi i})^2\oint_{\gamma_1}\frac{(1+s)^{n_2+k_2}}{s^{n_2+1}}\left\{\oint_{\gamma_2}\frac{(t+1)^{k_1}}{t^{n_1+1}}\cdot\prod_{j=1}^{n_2}\sum_{l_j=0}^{n_1}(\frac{(1+t)^{i_j}-1}{s})^{l_j}dt\right\}ds\\
=&\cdots\cdots
\end{aligned}
$$

Witten genus of string complete intersections

Proposition

Twisted Milnor hypersurfaces $H^{\mathbf{I}}_{n_1,n_2}(d_1, d_2)$ can't be string for $n_2 \geq 3$.

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Theorem

The $\widehat{A}$ genus of string complete intersections in generalised Bott manifold vanish.

Thus we shall seek the string complete intersections $H^{!}_{n_1,n_2}(d_1, d_2; d_3, d_4)$ Poincaré dual to $(d_1 u + d_2 v) \cdot (d_3 u + d_4 v) \in H^4(B_2; \mathbb{Z})$.

Witten genus

Consider Jocabi theta function

$$\theta(z,\tau) = 2q^{1/8} sin(\pi z) \prod_{j=1}^{\infty} [(1-q^j)(1-e^{2\pi iz}q^j)(1-e^{-2\pi iz}q^j)]$$

Let

$$f(x) = (e^{x/2} - e^{-x/2}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2},$$

$\frac{\theta(z)}{\theta'(0)} = f(2\pi iz)$, then Witten genus can be reformulated as

$$\begin{aligned}
\varphi_W(H^{\mathbf{I}}_{n_1,n_2}) &= \langle (\frac{u}{f(u)})^{n_1+1} \frac{v}{f(v)} \prod_{j=1}^{n_2} \frac{v-i_j u}{f(v-i_j u)} \frac{f(d_1 u + d_2 v)}{d_1 u + d_2 v} \frac{f(d_3 u + d_4 v)}{d_3 u + d_4 v}, [H^{\mathbf{I}}_{n_1,n_2}] \rangle \\
&= \langle (\frac{u}{f(u)})^{n_1+1} \frac{v}{f(v)} \prod_{j=1}^{n_2} \frac{v-i_j u}{f(v-i_j u)} f(d_1 u + d_2 v) \cdot f(d_3 u + d_4 v), [V] \rangle \\
&= \mathrm{Res}_0 \left\{ \mathrm{Res}_0 \frac{f(d_1 u + d_2 v) \cdot f(d_3 u + d_4 v)}{f^{n_1+1}(u) f(v) \prod_{j=1}^{n_2} f(v - i_j u)} du \right\} dv
\end{aligned}$$

Vanishing result

For string complete intersection $H^{\mathbf{I}}_{n_1,\ n_2}(d_1, d_2; d_3, d_4)$, if $\mathbf{I} = (i_1, \cdots, i_{n_2})$ are relative prime,

$$\varphi_W(H^{\mathbf{I}}_{n_1,\ n_2}(d_1, d_2; d_3, d_4)) = 0.$$

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It is more valuable to find examples with non vanishing Witten genus.

Toric forms in toric varieties

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Let $N \in \mathbb{R}^r$ be a lattice, M be its dual lattice. For a complete rational polyhedral fan $\Sigma \subset N \otimes \mathbb{R}$. A *degree function* $\deg : N \mapsto \mathbb{C}$ is a piecewise linear function that is linear on the cones of Σ.

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The **toric form** associated to $(N, \deg)$ is the function $f_{N,\deg} : \mathfrak{H} \longrightarrow \mathbb{C}$ defined by

$$f_{N,\deg}(q) = \sum_{m \in M} (\sum_{C \in \Sigma} (-1)^{codim\ C} a.c.(\sum_{n \in C} q^{m \cdot n} e^{2\pi i \deg(n)}))$$

here $q = e^{2\pi i \tau}$, $\tau \in \mathfrak{H}$, the upper halfplane, *a.c.* denotes analytic continuation.

Example

Let $N = \mathbb{Z}^2$, and Σ be the fan corresponding toric variety is projective plane $\mathbb{C}P^2$. Assume that deg takes α, β, γ on the generators $\mathbf{e}_1,\ \mathbf{e}_2,\ -\mathbf{e}_1 - \mathbf{e}_2$.

Example

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Then the toric form is

$$
\begin{aligned}
f_{N,\mathbf{deg}}(q) = \sum_{a,b\in\mathbb{Z}} & \frac{1}{(1-e^{2\pi i\alpha}q^{a})(1-e^{2\pi i\beta}q^{b})} + \frac{1}{(1-e^{2\pi i\beta}q^{b})(1-e^{2\pi i\gamma}q^{-a-b})} \\
& + \frac{1}{(1-e^{2\pi i\alpha}q^{a})(1-e^{2\pi i\gamma}q^{-a-b})} - \frac{1}{1-e^{2\pi i\alpha}q^{a}} - \frac{1}{1-e^{2\pi i\beta}q^{b}} - \frac{1}{1-e^{2\pi i\gamma}q^{-a-b}} + 1 \\
= \sum_{a,b\in\mathbb{Z}} & \frac{1-e^{2\pi i(\alpha+\beta+\gamma)}}{(1-e^{2\pi i\alpha}q^{a})(1-e^{2\pi i\beta}q^{b})(1-e^{2\pi i\gamma}q^{-a-b})}.
\end{aligned}
$$

Borisov, Gunnells gave toric form a topological interpretation by Hirzebruch-Riemann-Roch theorem.

Borisov, Gunnells

Assume that the toric variety X is nonsingular, and that $\alpha_i \notin \mathbb{Z}$ for all primitive generator of any 1-cone of Σ. Then

$$f_{N,\deg}(q) = \int_X \prod_i \frac{(D_i/2\pi i)\theta(D_i/2\pi i - \alpha_i)\theta'(0)}{\theta(D_i/2\pi i)\theta(-\alpha_i)}$$

where D_i denote the cohomology class of 1-cone of fan Σ.

Our "iterated residue" can be applied to calculate above Euler characteristic under certain conditions, thus give numerical identities regarding to toric forms.

Theorem

Consider toric variety $V = \mathbb{C}P(\eta^{\otimes i_1} \oplus \cdots \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}})$ over $\mathbb{C}P^{n_1}$, where η denote the tautological bundle of $\mathbb{C}P^{n_1}$.
If $(i_1, \cdots, i_{n_2})$ are relatively prime, $\sum_{j=0}^{n_2} \alpha_{n_1+2+j} \in \mathbb{Z}$, $\sum_{i=1}^{n_1+1} \alpha_i - \sum_{j=1}^{n_2} i_j \alpha_{n_1+2+j} \in \mathbb{Z}$, then $f_{N,\deg}(q) = 0$.

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We take **Hirzebruch surface** $F_k = \mathbb{C}P(\underline{\mathbb{C}} \oplus \mathcal{O}(k))$ over $\mathbb{C}P^1$ for example to illustrate our result. Its corresponding fan Σ consists of four two-dimensional cones generated by the pairs of vectors $(\mathbf{e}_1, \mathbf{e}_2),\ (\mathbf{e}_1, -\mathbf{e}_2),\ (-\mathbf{e}_1 + k\mathbf{e}_2, -\mathbf{e}_2),\ (-\mathbf{e}_1 + k\mathbf{e}_2, \mathbf{e}_2)$,

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We take **Hirzebruch surface** $F_k = \mathbb{C}P(\underline{\mathbb{C}} \oplus \mathcal{O}(k))$ over $\mathbb{C}P^1$ for example to illustrate our result. Its corresponding fan Σ consists of four two-dimensional cones generated by the pairs of vectors $(\mathbf{e}_1, \mathbf{e}_2)$, $(\mathbf{e}_1, -\mathbf{e}_2)$, $(-\mathbf{e}_1 + k\mathbf{e}_2, -\mathbf{e}_2)$, $(-\mathbf{e}_1 + k\mathbf{e}_2, \mathbf{e}_2)$,

Proposition

If $\alpha_2 + \alpha_4 \in \mathbb{Z}$, $\alpha_1 + \alpha_3 - k\alpha_4 \in \mathbb{Z}$, $\tau \in \mathfrak{H}$, $q = e^{2\pi i \tau}$,

$$f_{N,\deg}(q) = \sum_{a,b \in \mathbb{Z}} \frac{(1 - e^{2\pi i(\alpha_1 + \alpha_3)})(1 - q^{ka})}{(1 - e^{2\pi i \alpha_1} q^a)(1 - e^{2\pi i \alpha_3} q^{-a})(1 - e^{-2\pi i \alpha_4} q^b)(1 - e^{2\pi i \alpha_4} q^{ka-b})} = 0.$$

Thanks for listening!

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